AKLT Notes

Michael Ragone

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1 Recall: Representation Theory of $SU(2)$

Our goal is to understand the AKLT chain, a translation invariant spin-1 chain with a $SU(2)$-invariant nearest neighbor interaction. In this section we recall some representation theory of $SU(2)$, the special unitary group, and its corresponding Lie algebra $\mathfrak{su}_2$. This is effectively just tracing the arguments set forth in Appendix 9 of Nachtergaele and Sims' "Introduction to Quantum Spin Systems" with explicit computations.

1.1 The Spin-1 Irreducible Representation of $\mathfrak{su}(2)$

Recall that $D^{(s)}$ denotes the $(2s+1)$-irreducible representation of $\mathfrak{su}_2$, the corresponding Lie algebra of $SU(2)$, i.e. the spin-s representation. A central result proved in Nachtergaele in Sims, chapter 9.2 shows that for each dimension $n = 2s+1$, there is a unique irreducible representation of $\mathfrak{su}_2$ on $\mathbb{C}^n = D^{(s)}$. We need some material from the proof for the spin-1 case ($s = 1$) before proceeding with AKLT.

Recall the Pauli matrices, which form a convenient basis for the Lie algebra $\mathfrak{su}_2$:

$$\sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{1.1}$$

We can explicitly define the spin-1 Lie algebra representation $\pi : \mathfrak{su}_2 \to \mathrm{GL}(\mathbb{C}^3)$ by choosing the linear map π such that $\pi(\frac{1}{2}\sigma^j) = S^j, j = 1, 2, 3$ where

$$S^1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad S^2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \quad S^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{1.2}$$

and extend by linearity as per usual. As a sanity check, we note the commutation relations (verified by explicit computation), which up to scalars agree with the commutation relations we know and love for the Pauli spin-1/2 matrices:

$$\begin{aligned} [S^1, S^2] &= iS^3 \\ [S^2, S^3] &= iS^1 \\ [S^3, S^1] &= iS^2 \end{aligned} \tag{1.3}$$

Now, we define operators J_3, J_+, J_- on $\mathbb{C}^3$ as in Bruno's notes:

$$J_3 = S^3; \quad J_\pm = \frac{1}{\sqrt{2}} \left(S^1 \pm iS^2 \right) \tag{1.4}$$

or in matrices,

$$J_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}; \; J_+ = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}; \; J_- = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}; \tag{1.5}$$

which have the following commutation relations

$$[J_3, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = J_3 \tag{1.6}$$

The proof of irreducible representations of $\mathfrak{su}_2$ tells us that J_3 is diagonalizable over $\mathbb{C}^3$ with 3 distinct (and therefore simple) eigenvalues. Here, they are $\{1, 0, -1\}$ with corresponding eigenvectors the standard basis vectors in $\mathbb{C}^3$ labelled $\{v_1, v_0, v_{-1}\}$. Now, observe that as promised in the proof, J_+ and J_- "shift through" this eigenbasis in the following sense:

$$v_{-1} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \xmapsto{J_+(\cdot)} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \xmapsto{J_+(\cdot)} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \xmapsto{J_+(\cdot)} 0\,, \qquad v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \xmapsto{J_-(\cdot)} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \xmapsto{J_-(\cdot)} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \xmapsto{J_-(\cdot)} 0 \tag{1.7}$$

And indeed, any eigenvector v_λ with eigenvalue λ obeys for all $k \geq 0$,

$$J_3(J_\pm)^k v_\lambda = (\lambda \pm k)(J_\pm)^k v_\lambda \tag{1.8}$$

So, the highest weight subspace is spanned by the vector $v_1 = [1, 0, 0]^T$, since $v_1 \neq 0$ but $J_+ v_1 = 0$, and thus the highest weight is its eigenvalue, 1. The highest weight vector is unique, and we now have the data we need to study the tensor product representation $\widetilde{\pi}$, which we need to properly understand the AKLT interaction. Before proceeding, we derive a quick result regarding the Casimir operator.

1.2 The Casimir Element of a spin-s representation

The *Casimir operator* (also: *Casimir invariant* or *Casimir element*) for a spin-s Lie algebra representation with generators labelled S^1, S^2, S^3 and the spin vector $\mathbf{S} := (S^1, S^2, S^3)$ is the element

$$C := \mathbf{S}^2 = (S^1)^2 + (S^2)^2 + (S^3)^2 \tag{1.9}$$

The Casimir operator has a useful feature: $[\mathbf{S}^2, S^i] = 0$ for $i = 1, 2, 3$, and by linearity, this means that it commutes with every element in the algebra. To see that this is true, we demonstrate for S^1 and confidently assert that a similar computation holds for the other two. This follows from the commutation relations (1.3):

$$\begin{aligned}
[\mathbf{S}^2, S^1] = \sum_{j=1}^{3} [(S^j)^2, S^1] &= \sum_{j=1}^{3} S^j S^j S^1 + (-S^j S^1 S^j + S^j S^1 S^j) - S^1 S^j S^j \\
&= \sum_j S^j [S^j, S^1] + [S^j, S^1] S^j \\
&= \Big(S^2 [S^2, S^1] + [S^2, S^1] S^2\Big) + \Big(S^3 [S^3, S^1] + [S^3, S^1] S^3\Big) \\
&= -iS^2 S^3 - iS^3 S^2 + iS^3 S^2 + iS^2 S^3 \\
&= 0
\end{aligned} \tag{1.10}$$

Now, if our representation is irreducible, Schur's lemma gives us that since the Casimir operator commutes with every element in the algebra, it must be proportional to the identity $\mathbf{S}^2 = \lambda \mathbb{1}$. We can compute the constant lambda in terms of the spin s explicitly by defining $S_\pm$ similarly to $J_\pm$ (1.4) without the normalization constant:

$$S_\pm = S^1 \pm iS^2, \qquad [S_+, S_-] = 2S^3 \tag{1.11}$$

Now, $S_\pm = cJ_\pm$ for a scalar c, and thus they have the property that S_+ maps the highest weight eigenvector of S^3 with eigenvalue equal to the spin of the representation s, call it $|v\rangle$, to zero, so $S_+ |v\rangle = 0$. This means that

$$\begin{aligned}
0 = S_- S_+ |v\rangle &= (S_+ S_- - 2S^3) |v\rangle \\
&= \Big((S^1 + iS^2)(S^1 - iS^2) - 2S^3\Big) |v\rangle \\
&= \Big((S^1)^2 + (S^2)^2 + i(S^2 S^1 - S^1 S^2) - 2S^3\Big) |v\rangle \\
([S^1, S^2] = iS^3) \qquad &= \Big((S^1)^2 + (S^2)^2 + S^3 - 2S^3\Big) |v\rangle \\
&= \Big(\mathbf{S}^2 - (S^3)^2 - S^3\Big) |v\rangle \\
(S^3 |v\rangle = s |v\rangle \text{ and } \mathbf{S}^2 = \lambda \mathbb{1}) \qquad &= \lambda |v\rangle - s^2 |v\rangle - s |v\rangle
\end{aligned} \tag{1.12}$$

so after rearranging, $\lambda = s^2 + s = s(s+1)$. So, putting this all together, the Casimir operator for an irreducible spin-s representation $D^{(s)} = \mathbb{C}^{(2s+1)}$ has the following:

$$\mathbf{S}^2 = s(s+1)\mathbb{1} \tag{1.13}$$

As a note, the Casimir operator arises in quantum mechanics as the square of the length of the angular momentum vector (S^1, S^2, S^3). The fact that this commutes with each component of the angular momentum vector means that the squared angular momentum is simultaneously diagonalizable with each component separately. I'm told by trustworthy folks that this has important physical consequences.

1.3 Clebsch-Gordan Decomposition

We now do some explicit computations that will bring light to the AKLT interaction. In particular, we study the decomposition of two neighboring spin-1 sites, since that is the primary unit of study for the AKLT chain.

$$D^{(1)} \otimes D^{(1)} \cong D^{(0)} \oplus D^{(1)} \oplus D^{(2)} \tag{1.14}$$

This is a concrete computation that follows the more general argument in Nachtergaele and Sims 9.3.

We start furnished with the spin-1 irreducible representation $\pi : \mathfrak{su}_2 \to \mathbb{C}^3$. We wish to study the induced tensor product representation $\widetilde{\pi} : \mathfrak{su}_2 \to \mathbb{C}^3 \otimes \mathbb{C}^3$

$$\widetilde{\pi} = \pi \otimes \mathbb{1} + \mathbb{1} \otimes \pi \tag{1.15}$$

This representation is reducible–we wish to decompose it into a direct sum of irreducible representations. I.e., we will write $\mathbb{C}^3 \otimes \mathbb{C}^3$ as a direct sum of π-invariant subspaces such that π acts irreducibly on each subspace. These subspaces are what we refer to when we say "the spin-s subspace of $\mathbb{C}^3 \otimes \mathbb{C}^3$".

First, some notation building off of section 1.1. Define for $\widetilde{\pi} = \pi \otimes \mathbb{1} + \mathbb{1} \otimes \pi$

$$\widetilde{J}_3 = J_3 \otimes \mathbb{1} + \mathbb{1} \otimes J_3 \quad \text{and} \quad \widetilde{J}_\pm = J_\pm \otimes \mathbb{1} + \mathbb{1} \otimes J_\pm \tag{1.16}$$

The analogous commutation relations hold:

$$[\widetilde{J}_3, \widetilde{J}_\pm] = \pm \widetilde{J}_\pm, \quad [\widetilde{J}_+, \widetilde{J}_-] = \widetilde{J}_3 \tag{1.17}$$

Now, since we're working with two identical copies of $\mathbb{C}^3$, we can use our eigenbasis $\{v_1, v_0, v_{-1}\}$ from above our to make a basis for $\mathbb{C}^3 \otimes \mathbb{C}^3 : \{\Psi_{m,n} := v_m \otimes v_n \,|\, m,n = -1,0,1\}$. Observe that

$$\begin{aligned} \widetilde{J}_3 \Psi_{m,n} &= (J_3 \otimes \mathbb{1} + \mathbb{1} \otimes J_3) v_m \otimes v_n \\ &= J_3 v_m \otimes v_n + v_m \otimes J_3 v_n \\ &= m v_m \otimes v_n + v_m \otimes n v_n \\ &= (m+n)\Psi_{m,n} \end{aligned} \tag{1.18}$$

so in fact the $\Psi_{m,n}$ are eigenvectors of $\widetilde{J}_3$. Thanks to uniqueness of the highest weight vector v_1 for the representation π and that there is only one way to sum eigenvalues to get 2, the vector $\Psi_{1,1}$ is the unique highest weight vector of $\widetilde{\pi}$. So, we can repeatedly apply the lowering operator J_- to $\Psi_{1,1} = v_1 \otimes v_1$ to yield $2(2)+1 = 5$ vectors that are eigenvectors of $\widetilde{J}_3$ with eigenvalues $\{2, 1, 0, -1, -2\}$:

$$\begin{array}{c} v_1 \otimes v_1 \\ \downarrow \widetilde{J}_-(\cdot) \\ v_0 \otimes v_1 + v_1 \otimes v_0 \\ \downarrow \widetilde{J}_-(\cdot) \\ v_{-1} \otimes v_1 + 2v_0 \otimes v_0 + v_1 \otimes v_{-1} \\ \downarrow \widetilde{J}_-(\cdot) \\ 3v_{-1} \otimes v_0 + 3v_0 \otimes v_{-1} \\ \downarrow \widetilde{J}_-(\cdot) \\ 6v_{-1} \otimes v_{-1} \end{array} \tag{1.19}$$

Now, recall that $\widetilde{J}_3, \widetilde{J}_\pm$ form a basis for the image of $\mathfrak{su}_2$ for this representation. Since the above are all eigenvectors of $\widetilde{J}_3$, $\widetilde{J}_-(v_{-1} \otimes v_{-1}) = 0$, and $\widetilde{J}_+$ just shifts through this list of eigenvectors in the opposite direction, the subspace spanned by these eigenvectors is $\widetilde{\pi}$-invariant. Further, it is obviously irreducible since $\widetilde{J}_\pm$ shift throughout the entire space. Thus, this is a 5-dimensional representation of $\mathfrak{su}_2$, i.e. the spin-2 subspace $D^{(2)}$ of $\mathbb{C}^3 \otimes \mathbb{C}^3$.

Moving on, we see that since there are only two ways of adding $m + n = 1$ with $m, n = -1, 0, 1$, the dimension of the weight 1 eigenspace of $\widetilde{J}_3$ is 2. One vector $v_0 \otimes v_1 + v_1 \otimes v_0$ has already been accounted for in the spin-2 subspace, so we take another one. We choose the orthogonal vector $v_0 \otimes v_1 - v_1 \otimes v_0$. Applying the lowering operator as before,

$$\begin{array}{c} v_0 \otimes v_1 - v_1 \otimes v_0 \\ \downarrow \widetilde{J}_-(\cdot) \\ v_{-1} \otimes v_1 - v_1 \otimes v_{-1} \\ \downarrow \widetilde{J}_-(\cdot) \\ v_{-1} \otimes v_0 - v_0 \otimes v_{-1} \end{array} \tag{1.20}$$

and so we have our spin-1 subspace $D^{(1)}$. The final subspace will come from the weight 0 eigenspace of $\widetilde{J}_3$, so we may as well pick a vector orthogonal to the other two.

$$v_{-1} \otimes v_1 - v_0 \otimes v_0 + v_1 \otimes v_{-1} \tag{1.21}$$

which is our desired spin-0 subspace. This completes the Clebsch-Gordan decomposition we promised:

$$D^{(1)} \otimes D^{(1)} \cong D^{(2)} \oplus D^{(1)} \oplus D^{(0)} \tag{1.22}$$

and now that we have explicit bases for these spaces, any computation we might need to do can be easily performed.

Let's record this decomposition in one convenient spot, switching to bra-ket notation, so

$$v_{-1} = |-1\rangle, v_0 = |0\rangle, v_1 = |1\rangle; \quad v_i \otimes v_j = |i, j\rangle \tag{1.23}$$

We have the following basis for $D^{(1)} \otimes D^{(1)} \cong D^{(2)} \oplus D^{(1)} \oplus D^{(0)}$ (ignoring normalization and suppressing the underlying $D^{(1)} \otimes D^{(1)}$ so we just write $|1, 1; x, y\rangle = |x, y\rangle$)

$$\begin{aligned} &\text{Spin-2 } D^{(2)} : \{|1,1\rangle, \ \ |0,1\rangle + |1,0\rangle, \ \ |-1,1\rangle + 2\,|0,0\rangle + |1,-1\rangle, \\ &\qquad\qquad 3\,|-1,0\rangle + 3\,|0,-1\rangle, \ \ 6\,|-1,-1\rangle\} \\ &\text{Triplet } D^{(1)} : \{|0,1\rangle - |1,0\rangle, \ \ |-1,1\rangle - |1,-1\rangle, \ \ |-1,0\rangle - |0,-1\rangle\} \\ &\text{Singlet } D^{(0)} : \{|-1,1\rangle - |0,0\rangle + |1,-1\rangle\} \end{aligned} \tag{1.24}$$

2 AKLT

2.1 AKLT Chain: "Simple" exercise to see $h^{AKLT} = P^{(2)}$

The AKLT chain is a 1-dimensional chain of quantum spin-1 sites with underlying Hilbert space:

$$\mathcal{H} := \bigotimes_{x \in \mathbb{Z}} \mathbb{C}^3_x \tag{2.1}$$

so each site $x \in \mathbb{Z}$ has a copy of spin-1, $D^{(1)} \cong \mathbb{C}^3$. The AKLT interaction is a nearest neighbor interaction between two sites $x, x+1$, which we can describe as an operator on $\mathbb{C}^3 \otimes \mathbb{C}^3$:

$$h^{AKLT} := \frac{1}{3}\mathbb{1} + \frac{1}{2}\mathbf{S}_x \cdot \mathbf{S}_{x+1} + \frac{1}{6}(\mathbf{S}_x \cdot \mathbf{S}_{x+1})^2 \tag{2.2}$$

where $\mathbf{S}_x = (S_x^1, S_x^2, S_x^3)$ is the vector of the 3×3 spin-1 matrices acting at the xth site of the chain. Using the decomposition from the previous section, it will be shown that we can write the AKLT interaction as an orthogonal projection $P^{(2)}_{x,x+1}$ on the spin-2 $D^{(2)}$ subspace of $\mathbb{C}^3 \otimes \mathbb{C}^3$ (this was further verified by a short matlab script written by the author).

We define the *Casimir operator* C for this representation of $\mathfrak{su}_2$, which we will use to verify that the AKLT interaction is a projection map:

$$\begin{aligned} C &= (S^1 \otimes \mathbb{1} + \mathbb{1} \otimes S^1)^2 + (S^2 \otimes \mathbb{1} + \mathbb{1} \otimes S^2)^2 + (S^3 \otimes \mathbb{1} + \mathbb{1} \otimes S^3)^2 \\ &= \left((S^1)^2 + (S^2)^2 + (S^3)^2\right) \otimes \mathbb{1} + \mathbb{1} \otimes \left((S^1)^2 + (S^2)^2 + (S^3)^2\right) \\ &\quad + 2\left(S^1 \otimes S^1 + S^2 \otimes S^2 + S^3 \otimes S^3\right) \\ &= \mathbf{S}^2 \otimes \mathbb{1} + \mathbb{1} \otimes \mathbf{S}^2 + 2h \end{aligned} \tag{2.3}$$

where $h = S^1 \otimes S^1 + S^2 \otimes S^2 + S^3 \otimes S^3$. Looking back at (1.10), where we proved $[\mathbf{S}^2, S^i] = 0$ for $i = 1, 2, 3$, it's not too hard to see that the Casimir element commutes with the generators $S^i \otimes \mathbb{1} + \mathbb{1} \otimes S^i$ of the representation on $\mathbb{C}^3 \otimes \mathbb{C}^3$. Similarly to as in section 1.2, we can then apply Schur's lemma to the operator restricted to each irreducible subspace in the decomposition $D^{(0)} \oplus D^{(1)} \oplus D^{(2)}$ to see that we can write C as a sum of orthogonal projections onto each spin subspace:

$$C = \lambda_0 P^{(0)} + \lambda_1 P^{(1)} + \lambda_2 P^{(2)} \tag{2.4}$$

To compute the λ_s, we give thanks to computation (1.13), which tells us each λ_s is dictated by the spin of its corresponding subspace. In particular, $\lambda_s = s(s+1)$. So

$$C = 2P^{(1)} + 6P^{(2)} \tag{2.5}$$

Now, on either spin-1 factor in $D^{(1)} \otimes D^{(1)}$, the operator $\mathbf{S}^2$ is the Casimir operator and thus $\mathbf{S}^2 = 2\mathbb{1}$. But since C and $\mathbf{S}^2 \otimes \mathbb{1}$ and $\mathbb{1} \otimes \mathbf{S}^2$ all commute, they are simultaneously diagonalizable, and we can safely write

$$2P^{(1)} + 6P^{(2)} = C = (2\mathbb{1}) \otimes \mathbb{1} + \mathbb{1} \otimes (2\mathbb{1}) + 2h = 4\mathbb{1} + 2h \tag{2.6}$$

So doing algebra tells us that

$$\begin{aligned} h &= \frac{1}{2}(2P^{(1)} + 6P^{(2)} - 4(P^{(0)} + P^{(1)} + P^{(2)})) \\ &= -2P^{(0)} - P^{(1)} + P^{(2)} \end{aligned} \tag{2.7}$$

Let's return to the AKLT interaction, now that we have more useful expression for h. We can use the fact that these are orthogonal projections to write

$$\begin{aligned} h^{AKLT} &= \frac{1}{3}\mathbb{1} + \frac{1}{2}h + \frac{1}{6}h^2 \\ &= (\frac{1}{3} - 1 + \frac{4}{6})P^{(0)} \\ &\quad + (\frac{1}{3} - \frac{1}{2} + \frac{1}{6})P^{(1)} \\ &\quad + (\frac{1}{3} + \frac{1}{2} + \frac{1}{6})P^{(2)} \\ &= P^{(2)} \end{aligned} \tag{2.8}$$

So, as desired, the AKLT interaction is the orthogonal projection onto the spin-2 subspace.

2.2 AKLT Chain: $D^{(1)} \otimes D^{(1/2)} \cong D^{(1/2)} \oplus D^{(3/2)}$ and the intertwining map V

Now, we graduate to the AKLT chain of length n. Its Hamiltonian is given by a sum of the earlier nearest neighbor interactions:

$$H_{[1,n]} = \sum_{x=1}^{n} \frac{1}{3}\mathbb{1} + \frac{1}{2}\mathbf{S}_x \cdot \mathbf{S}_{x+1} + \frac{1}{6}(\mathbf{S}_x \cdot \mathbf{S}_{x+1})^2 \tag{2.9}$$

We saw that the nearest-neighbor AKLT interaction could be written as the orthogonal projection onto the spin-2 subspace $D^{(2)} \subseteq \mathbb{C}^3 \otimes \mathbb{C}^3$. Its ground state space is then the kernel of that projection operator, which is the sum of the spin-0 and spin-1 subspaces: $D^{(0)} \oplus D^{(1)} \subseteq \mathbb{C}^3 \oplus \mathbb{C}^3$. It turns out the the AKLT ground state for a chain of length n is in fact also 4-dimensional, just as in the case $n = 2$. To see this, we need to look at another Clebsch-Gordan decomposition: $D^{(1)} \otimes D^{(1/2)} \cong D^{(1/2)} \oplus D^{(3/2)}$. We will not compute the explicit bases for this one (feel free to ask the author, he has them written up).

This decomposition furnishes us with an isometric embedding from the spin-1/2 subspace to the tensor product $V : \mathbb{C}^2 \to \mathbb{C}^3 \otimes \mathbb{C}^2$, such that

$$VD^{(1/2)}(g) = (D^{(1)}(g) \otimes D^{(1/2)}(g))V\,, \quad \text{for all } g \in SU(2) \tag{2.10}$$

We say that V *intertwines* the $SU(2)$ representations $D^{(1/2)}$ and $D^{(1)} \otimes D^{(1/2)}$. Put another way, the following diagram commutes for all $g \in SU(2)$:

$$\begin{array}{ccc} \mathbb{C}^2 & \xrightarrow{V} & \mathbb{C}^3 \otimes \mathbb{C}^2 \\ {\scriptstyle D^{(1/2)}(g)}\big\downarrow & & \big\downarrow{\scriptstyle D^{(1)}(g)\otimes D^{(1/2)}(g)} \\ \mathbb{C}^2 & \xrightarrow[V]{} & \mathbb{C}^3 \otimes \mathbb{C}^2 \end{array} \tag{2.11}$$

If we choose the orthogonal eigenvector basis of $\mathbb{C}^2$ given by the third spin operator, $\{|1/2; -1/2\rangle, |1/2; 1/2\rangle\}$ and repeat the computations in 1.3, we can get matrix elements of the intertwiner V, appropriately called Clebsch-Gordan coefficients:

$$V|1/2; m\rangle = \sum_{\substack{m_1, m_2 \\ m_1 + m_2 = m}} \Big(\langle 1, 1/2; m_1, m_2|\,|1/2; m\rangle\Big)\, |1, 1/2; m_1, m_2\rangle \tag{2.12}$$

We will use the intertwiner map to define Matrix Product States. In fact, the ground state of the AKLT chain will be a Matrix Product State, which is great news for us.

2.3 AKLT Chain: Matrix Product States (MPS)

Now, let $n \geq 2$. Let $\alpha, \beta \in \mathbb{C}^2$, and define $\psi^{(n)}_{\alpha\beta} \in \mathcal{H}_{[1,n]}$ by

$$\psi^{(n)}_{\alpha\beta} = (\mathbb{1}_3^{\otimes n} \otimes \langle\beta|)(\underbrace{\mathbb{1}_3 \otimes \cdots \otimes \mathbb{1}_3}_{n-1} \otimes V) \ldots (\mathbb{1} \otimes V)V|\alpha\rangle \tag{2.13}$$

Notice the sequence of embeddings this product defines:

$$\begin{array}{c} |a\rangle \longmapsto V|a\rangle \longmapsto (\mathbb{1}_3 \otimes V)V|a\rangle \longmapsto (\mathbb{1}_3 \otimes \mathbb{1}_3 \otimes V)(\mathbb{1}_3 \otimes V)V|a\rangle \longmapsto \ldots \\ \mathbb{C}^2 \hookrightarrow \mathbb{C}^3 \otimes (\mathbb{C}^2) \hookrightarrow \mathbb{C}^3 \otimes (\mathbb{C}^3 \otimes \mathbb{C}^2) \hookrightarrow \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes (\mathbb{C}^3 \otimes \mathbb{C}^2) \hookrightarrow \ldots \end{array} \tag{2.14}$$

until at the very end, the straggler copy of $\mathbb{C}^2$ is killed off by the $\langle\beta|$, yielding a vector $\psi^{(n)}_{\alpha\beta} \in (\mathbb{C}^3)^{\otimes n}$. Let's do a quick example with $n = 1$ of computation (11.6) in the notes. Let $\psi^{(1)}_{\alpha\beta} = \psi_{\alpha\beta}$. Note when we write $D^{(1)}$, we really mean $D^{(1)}(g)$, where $g \in SU(2)$. After suppressing the g, the intertwiner property means

$VD^{(1/2)} = (D^{(1)} \otimes D^{(1/2)})V$.

$$\begin{aligned}
D^{(1)}\psi_{\alpha\beta} &= D^{(1)}(\mathbb{1}_3 \otimes \langle\beta|)V|\alpha\rangle \\
(a) \qquad &= D^{(1)}(\mathbb{1}_3 \otimes \langle\beta|)\left(\mathbb{1}_3 \otimes (D^{(1/2)})^* D^{(1/2)}\right) V|\alpha\rangle \\
&= D^{(1)}(\mathbb{1}_3 \otimes \left\langle D^{(1/2)}\beta\right|)\left(\mathbb{1}_3 \otimes D^{(1/2)}\right) V|\alpha\rangle \\
(b) \qquad &= (\mathbb{1}_3 \otimes \left\langle D^{(1/2)}\beta\right|)\left(D^{(1)} \otimes D^{(1/2)}\right) V|\alpha\rangle \\
(c) \qquad &= (\mathbb{1}_3 \otimes \left\langle D^{(1/2)}\beta\right|)VD^{(1/2)}|\alpha\rangle \\
&= (\mathbb{1}_3 \otimes \left\langle D^{(1/2)}\beta\right|)V\left|D^{(1/2)}\alpha\right\rangle
\end{aligned} \tag{2.15}$$

where (a) holds since $D^{(1/2)}$ is a unitary representation (just check Pauli matrices) and thus $(D^{(1/2)})^* D^{(1/2)} = \mathbb{1}_2$, (b) holds by linear algebra (check below), and (c) holds by the intertwining property.

To see (b), let $e_i \in \mathbb{C}^3$ and $f_j \in \mathbb{C}^2$ be bases, let $\beta \in \mathbb{C}^2$, $A \in M_3(\mathbb{C})$, and $\alpha \in \mathbb{C}^3 \otimes \mathbb{C}^2$. Then

$$\begin{aligned}
A(\mathbb{1}_3 \otimes \langle\beta|)\left(\sum_{i,j} \alpha_{ij}|e_i\rangle \otimes |f_j\rangle\right) &= A\left(\sum_{i,j} \alpha_{ij}|e_i\rangle \otimes \langle\beta|\,|f_j\rangle\right) \\
&= \sum_{i,j} \langle\beta, f_j\rangle \alpha_{ij} A|e_i\rangle
\end{aligned} \tag{2.16}$$

Now to the general n case. So, using the intertwining property of V n times, we see

$$\begin{aligned}
(D^{(1)})^{\otimes n}\psi^{(n)}_{\alpha\beta} &= (\mathbb{1}_3^{\otimes n} \otimes \left\langle D^{(1/2)}\beta\right|)(\underbrace{D^{(1)} \otimes \cdots \otimes D^{(1)}}_{n} \otimes D^{(1/2)}V)\ldots V|\alpha\rangle \\
&= (\mathbb{1}_3^{\otimes n} \otimes \left\langle D^{(1/2)}\beta\right|)(\underbrace{\mathbb{1}_3 \otimes \cdots \otimes \mathbb{1}_3}_{n-1} \otimes V)\ldots(\mathbb{1}_3 \otimes V)V\left|D^{(1/2)}\alpha\right\rangle
\end{aligned} \tag{2.17}$$

This means that $SU(2)$ acts on the space $\{\psi^{(n)}_{\alpha\beta} \,|\, \alpha, \beta \in \mathbb{C}^2\}$ by the representation $(D^{(1/2)})^* \otimes D^{(1/2)}$. Now, we have as a fact from representation theory that a representation Π is irreducible if and only if its dual representation Π^* is irreducible. Then since the dual representation $(D^{(1/2)})^*$ is an irreducible representation of $SU(2)$ into a vector space the same dimension as $D^{(1/2)}$, we can leverage uniqueness of irreducible representations of $SU(2)$ to see that $D^{(1/2)} \cong (D^{(1/2)})^*$, and thus $SU(2)$ acts on the space $\{\psi^{(n)}_{\alpha,\beta}\}$ by

$$(D^{(1/2)})^* \otimes D^{(1/2)} \cong D^{(1/2)} \otimes D^{(1/2)} \cong D^{(0)} \oplus D^{(1)} \tag{2.18}$$

which in particular proves

$$P^{(2)}\psi^{(2)}_{\alpha\beta} = 0 \tag{2.19}$$

for all $\alpha, \beta \in \mathbb{C}^2$. So these states $\psi^{(2)}_{\alpha\beta}$ are ground states of the finite AKLT chain, and this will soon allow us to show that the states $\psi^{(n)}_{\alpha\beta}$ are ground states of the finite AKLT chain. We will now derive the MPS representation of the states $\psi^{(n)}_{\alpha\beta}$ and show in Prop 11.1 that

$$\text{span}\{\psi^{(n)}_{\alpha\beta} \,|\, \alpha, \beta \in \mathbb{C}^2\} = \ker H_{[1,n]} \tag{2.20}$$

It will be useful to have an explicit matrix form of the intertwiner map V defined in (2.10). Given the standard basis $|1\rangle, |0\rangle, |-1\rangle$ of $\mathbb{C}^3$, we can define 2×2 matrices $v_i, i = 1, 0, -1$ by

$$V|\alpha\rangle = \sum_i |i\rangle \otimes v_i|\alpha\rangle \tag{2.21}$$

Explicitly:

$$V = \begin{bmatrix} v_1 \\ v_0 \\ v_{-1} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \sqrt{\frac{2}{3}} & 0 \\ -\sqrt{\frac{1}{3}} & 0 \\ 0 & \sqrt{\frac{1}{3}} \\ 0 & -\sqrt{\frac{2}{3}} \\ 0 & 0 \end{bmatrix} \tag{2.22}$$

As a sanity check, we can compute $V^*V = \mathbb{1}_2$, confirming that it is indeed an isometric embedding of $\mathbb{C}^2$ into $\mathbb{C}^3 \otimes \mathbb{C}^2$. We now check claim (11.10) in the notes. First for $n = 1$.

$$\begin{aligned} \left\langle i_1 | \psi^{(1)}_{\alpha\beta} \right\rangle &= \left\langle i_1 \right| \left((\mathbb{1} \otimes \langle\beta|) V \left|\alpha\right\rangle \right) \rangle \\ &= \left\langle i_1 \right| \left((\mathbb{1} \otimes \langle\beta|) \sum_{j_1} |j_1\rangle \otimes v_{j_1} |\alpha\rangle \right) \Big\rangle \\ &= \left\langle i_1 \right| \left(\sum_{j_1} |j_1\rangle \otimes \langle\beta| v_{j_1} |\alpha\rangle \right) \Big\rangle \\ \textit{(orthonormality)} \quad &= \langle\beta| v_{i_1} |\alpha\rangle \\ \textit{(cyclicity of trace)} \quad &= \operatorname{Tr} |\alpha\rangle \langle\beta| v_{i_1} \end{aligned} \tag{2.23}$$

Now, for $n = 2$. Just to convince ourselves (the induction will be obvious). Recall the notation that $|j_1, j_2\rangle := |j_1\rangle \otimes |j_2\rangle$.

$$\begin{aligned} \left\langle i_1, i_2 | \psi^{(2)}_{\alpha\beta} \right\rangle &= \left\langle i_1, i_2 \right| \left((\mathbb{1} \otimes \mathbb{1} \otimes \langle\beta|)(\mathbb{1} \otimes V) V |\alpha\rangle \right) \rangle \\ &= \left\langle i_1, i_2 \right| \left((\mathbb{1} \otimes \mathbb{1} \otimes \langle\beta|)(\mathbb{1} \otimes V) \sum_{j_1} |j_1\rangle \otimes v_{j_1} |\alpha\rangle \right) \Big\rangle \\ &= \left\langle i_1, i_2 \right| \left((\mathbb{1} \otimes \mathbb{1} \otimes \langle\beta|) \sum_{j_1} |j_1\rangle \otimes V v_{j_1} |\alpha\rangle \right) \Big\rangle \\ &= \left\langle i_1, i_2 \right| \left((\mathbb{1} \otimes \mathbb{1} \otimes \langle\beta|) \sum_{j_1} |j_1\rangle \otimes \left(\sum_{j_2} |j_2\rangle \otimes v_{j_2} v_{j_1} |\alpha\rangle \right) \right) \Big\rangle \\ &= \left\langle i_1, i_2 \right| \left(\sum_{j_1, j_2} |j_1\rangle \otimes |j_2\rangle \otimes \langle\beta| v_{j_2} v_{j_1} |\alpha\rangle \right) \Big\rangle \\ \textit{(orthonormality)} \quad &= \langle\beta| v_{i_2} v_{i_1} |\alpha\rangle \\ \textit{(cyclicity of trace)} \quad &= \operatorname{Tr} |\alpha\rangle \langle\beta| v_{i_2} v_{i_1} \end{aligned} \tag{2.24}$$

So, we can rest assured that the formula (11.10) for arbitrary n is correct:

$$\left\langle i_1, \dots, i_n | \psi^{(n)}_{\alpha\beta} \right\rangle = \langle\beta| v_{i_n} \dots v_{i_1} |\alpha\rangle = \operatorname{Tr} |\alpha\rangle \langle\beta| v_{i_n} \dots v_{i_1} \tag{2.25}$$

Which, in other words (since $\{|i_1, \dots, i_n\rangle \mid i_j = -1, 0, 1\}$ forms a basis for $\bigotimes_{j=1}^n D^{(1)}$, the Hilbert space of the AKLT spin-1 chain of length n),

$$\psi^{(n)}_{\alpha\beta} = \sum_{i_1, \dots, i_n} \operatorname{Tr}[|\alpha\rangle \langle\beta| v_{i_n} \dots v_{i_1}] \, |i_1, \dots, i_n\rangle \tag{2.26}$$

Now, since the rank 1 maps $\{|\alpha\rangle\langle\beta| \mid \alpha, \beta \in \mathbb{C}^2\}$ form a basis for the 2×2 complex matrices M_2, we can extend the vectors $\psi^{(n)}$ linearly to a map $\psi^{(n)} : M_2 \to \mathcal{H}_{[1,n]}$: to see a more familiar definition of Matrix Product States:

$$\psi^{(n)}(B) = \sum_{i_1,\ldots,i_n} \mathrm{Tr}[Bv_{i_n} \ldots v_{i_1}] \, |i_1, \ldots i_n\rangle \,, \quad B \in M_2 \tag{2.27}$$

And of course, this gives us

$$\mathrm{span}\{\psi^{(n)}_{\alpha\beta} \mid \alpha, \beta \in \mathbb{C}^2\} = \{\psi^{(n)}(B) \mid B \in M_2\} \tag{2.28}$$

and we define $\mathcal{G}_n = \{\psi^{(n)}(B) \mid B \in M_2\}$. We can now prove the inclusion

$$\mathcal{G}_n \subseteq \ker H_{[1,n]} \tag{2.29}$$

by a direct computation. Let $x = 1, \ldots, n-1$, and compute the expectation of $P^{(2)}_{x,x+1} \in_{[1,n]}$ in a state $\psi^{(n)}(B)$ (recall that $H_{[1,n]} = \sum_{x=1}^{n-1} P^{(2)}_{x,x+1}$, and $\delta_{i,j}$ is the kronecker delta)

$$\begin{aligned}
&\langle \psi^{(n)}(B), P^{(2)}_{x,x+1}\psi^{(n)}(B) \\
&= \sum_{\substack{i_1,\ldots,i_n \\ j_1,\ldots,j_n}} \overline{\mathrm{Tr}[Bv_{i_n} \ldots v_{i_1}]}\mathrm{Tr}[Bv_{j_n} \ldots v_{j_1}] \left(\langle i_1, \ldots, i_n | \, \mathbb{1}_{[1,x-1]} \otimes P^{(2)} \otimes \mathbb{1}_{[x+2,n]} \, |j_1, \ldots, j_n\rangle \right) \\
&= \sum_{\substack{i_1,\ldots,i_n \\ j_1,\ldots,j_n}} \overline{\mathrm{Tr}[Bv_{i_n} \ldots v_{i_1}]}\mathrm{Tr}[Bv_{j_n} \ldots v_{j_1}]\delta_{i_1,j_1} \ldots \delta_{i_{x-1},j_{x-1}}(\langle i_x, i_{x+1}| \, P^{(2)}_{x,x+1} \, |j_x, j_{x+1}\rangle)\delta_{i_{x+2},j_{x+2}} \ldots \delta_{i_n,j_n} \\
&= \sum_{\substack{i_1,\ldots,i_{x-1} \\ i_{x+2},\ldots,i_n}} \sum_{\substack{i_x,i_{x+1} \\ j_x,j_{x+1}}} \overline{\mathrm{Tr}[Bv_{i_n} \ldots v_{i_1}]}\mathrm{Tr}[Bv_{j_n} \ldots v_{j_1}] \, \langle i_x, i_{x+1}| \, P^{(2)}_{x,x+1} \, |j_x, j_{x+1}\rangle \\
&= \sum_{\substack{i_1,\ldots,i_{x-1} \\ i_{x+2},\ldots,i_n}} \sum_{\substack{i_x,i_{x+1} \\ j_x,j_{x+1}}} \Bigg(\overline{\mathrm{Tr}[(v_{i_{x-1}} \ldots v_1 Bv_{i_n} \ldots v_{i_{x+2}})v_{i_{x+1}}v_{i_x}]}\mathrm{Tr}[(v_{i_{x-1}} \ldots v_1 Bv_{i_n} \ldots v_{i_{x+2}})v_{j_{x+1}}v_{j_x}] \\
&\qquad \langle i_x, i_{x+1}| \, P^{(2)}_{x,x+1} \, |j_x, j_{x+1}\rangle \Bigg) \\
&= \sum_{\substack{i_1,\ldots,i_{x-1} \\ i_{x+2},\ldots,i_n}} \left\langle \psi^{(2)}(v_{i_{x-1}} \ldots v_1 Bv_{i_n} \ldots v_{i_{x+2}}), P^{(2)}\psi^{(2)}(v_{i_{x-1}} \ldots v_1 Bv_{i_n} \ldots v_{i_{x+2}}) \right\rangle
\end{aligned} \tag{2.30}$$

And now, since $C = v_{i_{x-1}} \ldots v_1 Bv_{i_n} \ldots v_{i_{x+2}} \in M_2$, using equation (2.19) and the linear extension from the basis vectors (2.27) guarantees that $P^{(2)}\psi^{(2)}(C) = 0$. Thus each term in the final sum is 0, so the above expectation is indeed 0, proving that indeed $\mathcal{G}_n \subseteq \ker H_{[1,n]}$. We now prove the other inclusion to get the following proposition, which establishes that the ground state space of the finite AKLT is in fact the space of matrix product states. It will later turn out that the infinite chain has a unique zero-energy ground state, but this will require some work on correlation functions in the thermodynamic limit.

2.4 Prop 11.1: Ground State Space of Finite AKLT Chain = MPS Space

Proposition 11.1, Nachtergaele and Sims: For all $n \geq 2$, we have

$$\ker H_{[1,n]} = \{\psi^{(n)}(B) \mid B \in M_2\} \tag{2.31}$$

Proof:

2.4.1 Proving $\dim\{\psi^{(n)}\} = 4$

So far, we have shown $\mathcal{G}_n \subseteq \ker H_{[1,n]}$, and we now need to show the opposite containment.

We start by showing that

$$\dim\{\psi^{(2)}(B) \mid B \in M_2\} = 4 \tag{2.32}$$

To see this, recall the action of $SU(2)$ on the MPS vectors (2.18): it (here, $D^{(1)} \otimes D^{(1)}$) acts on the space $\{\psi^{(2)}(B)\}$ by $(D^{(1/2)})^* \otimes D^{(1/2)} \cong D^{(0)} \oplus D^{(1)}$:

$$(D^{(1)} \otimes D^{(1)})\psi^{(2)}(B) = \psi^{(2)}((D^{(1/2)})^* B D^{(1/2)}) \tag{2.33}$$

Either these vectors vanish, or $\psi^{(2)}(\mathbb{1})$ is a singlet and $\psi^{(2)}(\sigma^i), i = 1, 2, 3$ is a triplet representation of $SU(2)$. We will show in a moment that these vectors are nonzero and in fact, mutually orthogonal, but first let's see why $\psi^{(2)}(\mathbb{1})$ is a singlet. Recall that we can use the Clebsch-Gordan decomposition (1.3) to compute a good basis for $D^{(1)} \oplus D^{(0)}$. In particular, look at the handy-dandy decomposition (1.24) that identifies the Spin-2 subspace, the Triplet subspace, and the Singlet subspace. We can compute where $\psi^{(2)}$ maps the identity matrix and verify that it is indeed a singlet (meaning it has 0 projection in $D^{(2)}$ and $D^{(1)}$, which the author checked)

$$\begin{aligned} \psi^{(2)}(\mathbb{1}) &= \mathrm{Tr}(v_{-1}v_1)\,|1,-1\rangle + \mathrm{Tr}(v_0^2)\,|0,0\rangle + \mathrm{Tr}(v_1 v_{-1})\,|-1,1\rangle \\ &= -\frac{2}{3}(|1,-1\rangle - |0,0\rangle + |-1,1\rangle) \end{aligned} \tag{2.34}$$

Likewise, we can check for the Pauli matrices that $\psi^{(2)}$ maps them to triplet vectors. Let's verify this for the first Pauli matrix (again, the other parts have 0 trace):

$$\begin{aligned} \psi^{(2)}(\sigma^1) &= \mathrm{Tr}(\sigma^1 v_1 v_0)\,|0,1\rangle + \mathrm{Tr}(\sigma^1 v_0 v_1)\,|1,0\rangle \\ &\quad + \mathrm{Tr}(\sigma^1 v_1 v_{-1})\,|-1,1\rangle + \mathrm{Tr}(\sigma^1 v_{-1} v_1)\,|1,-1\rangle \\ &\quad + \mathrm{Tr}(\sigma^1 v_0 v_{-1})\,|-1,0\rangle + \mathrm{Tr}(\sigma^1 v_{-1} v_0)\,|0,-1\rangle \\ &= \frac{i}{3\sqrt{2}}\big(|0,1\rangle - |1,0\rangle\big) \\ &\quad + 0 + 0 \\ &\quad + \frac{-i}{3\sqrt{2}}\big(|-1,0\rangle - |0,-1\rangle\big) \end{aligned} \tag{2.35}$$

So as desired, the first Pauli matrix maps to a nonzero triplet. This similarly holds for the other two Pauli matrices: we record the concrete computations for posterity.

$$\begin{aligned} \psi^{(2)}(\sigma^1) &= \frac{i}{3\sqrt{2}}\big(|0,1\rangle - |1,0\rangle\big) + \frac{-i}{3\sqrt{2}}\big(|-1,0\rangle - |0,-1\rangle\big) \\ \psi^{(2)}(\sigma^2) &= \frac{-1}{3\sqrt{2}}\big(|0,1\rangle - |1,0\rangle\big) + \frac{-1}{3\sqrt{2}}\big(|-1,0\rangle - |0,-1\rangle\big) \\ \psi^{(2)}(\sigma^3) &= \frac{i}{3}\big(|-1,1\rangle - |1,-1\rangle\big) \end{aligned} \tag{2.36}$$

So the map $B \mapsto \psi^{(2)}(B)$ is in fact injective. We can now use induction on n to show that the maps $\psi^{(n)} : M_2 \to \mathcal{H}_{[1,n]}$ are injective for $n \geq 2$. To do this, note that if $\psi^{(n)}$ is injective, there exists a constant $c_n > 0$ such that $\left\|\psi^{(n)}(B)\right\|^2 \geq c_n \mathrm{Tr} B^* B$. Let's quickly prove why this is true, then return to the proof.

Lemma: If $\psi : M_k \to \mathbb{C}^m$ is an injective linear map, then there exists a constant $c > 0$ such that

$$\left\|\psi(B)\right\|^2 \geq c\mathrm{Tr}B^*B \quad \text{for all } B \in M_k \tag{2.37}$$

Proof of Lemma : Note that with the Hilbert-Schmidt inner product, M_k forms a Hilbert space.

$$\langle A, B\rangle_{HS} := \mathrm{Tr}A^*B \tag{2.38}$$

Now, take an orthonormal basis for M_k $\{\beta_1, \dots, \beta_k\}$. Since ψ is an injective map, we have that for each β_i, there exists $c_i > 0$ such that

$$\left\|\psi(\beta_i)\right\|^2 \geq c_i\|\beta_i\|_{HS}^2 = c_i\mathrm{Tr}\beta_i^*\beta_i \tag{2.39}$$

Let $c := \min_i c_i$. Then, we can write any $B = \sum_j b_j\beta_j$, and we have by orthonormality

$$\begin{aligned}\left\|\psi(B)\right\|^2 = \left\|\psi(\sum_j(b_j\beta_j))\right\|^2 = \sum_j\left|b_j\right|^2\left\|\psi(\beta_j)\right\|^2 &\geq \sum_j\left|b_j\right|^2 c\|\beta_i\|_{HS}^2 \\ &= c\|B\|_{HS}^2 \\ &= c\mathrm{Tr}B^*B\end{aligned} \tag{2.40}$$

□

Lemma in hand, we return to the proof. Using the definition of matrix product states, we can estimate $\left\|\psi^{(n+1)}(B)\right\|$:

$$\begin{aligned}\left\|\psi^{(n+1)}(B)\right\|^2 &= \left\|\sum_{i_1,\dots,i_n}\mathrm{Tr}[Bv_{i_{n+1}}v_{i_n}\dots v_{i_1}]\,|i_1,\dots,i_n,i_{n+1}\rangle\right\|^2 \\ &= \sum_{i_{n+1}}\sum_{i_1,\dots,i_n}\left|\mathrm{Tr}Bv_{i_{n+1}}v_{i_n}\dots v_{i_1}\right|^2 \\ &= \sum_{i_{n+1}}\left\|\psi^{(n)}(Bv_{i_{n+1}})\right\|^2 \\ &\geq c_n\sum_{i_{n+1}}\mathrm{Tr}(Bv_{i_{n+1}})^*Bv_{i_{n+1}} \\ &= c_n\sum_{i_{n+1}}\mathrm{Tr}(v_{i_{n+1}}v_{i_{n+1}}^*B^*B) \\ &= c_n\mathrm{Tr}\left[\sum_{i_{n+1}}v_{i_{n+1}}v_{i_{n+1}}^*\right]B^*B \\ &= c_n\mathrm{Tr}([\mathbb{1}]B^*B) \\ &= c_n\mathrm{Tr}B^*B\end{aligned} \tag{2.41}$$

Where the second to last line follows by computing the bracketed sum with the explicit form of the matrices v_i. Since $c_n > 0$ by induction hypothesis, we are done and $\psi^{(n+1)}$ is injective.

2.4.2 From Length 2 to Length 3

As the next step in the proof, we show an intersection property, which will later be souped up for length n chains.

$$\ker H_{[1,3]} = (\mathcal{G}_2 \otimes \mathbb{C}^3) \cap (\mathbb{C}^3 \otimes \mathcal{G}_2) = \mathcal{G}_3 \tag{2.42}$$

We have already shown the inclusion $\mathcal{G}_3 \subseteq \ker H_{[1,3]}$ before this proof began. We have just now shown by demonstrating injectivity of $\psi^{(n)} : M_2 \to \mathcal{H}_{[1,n]}$ that $\dim\ker H_{[1,3]} \geq 4$. If we now show that $\dim\ker H_{[1,3]} \leq 4$, we will have the desired $\ker H_{[1,3]} = \mathcal{G}_3$. Let us convince ourselves of the first equality before proceeding:

$$\ker H_{[1,3]} = (\mathcal{G}_2 \otimes \mathbb{C}^3) \cap (\mathbb{C}^3 \otimes \mathcal{G}_2) \tag{2.43}$$

Now, we have already shown that $\mathcal{G}_2 = \ker H_{[1,2]} = \ker H_{[2,3]}$. Taking these two nearest-neighbor Hamiltonians as acting on a chain with 3 sites, we have that

$$\ker H_{[1,2]} = \mathcal{G}_2 \otimes \mathbb{C}^3, \qquad \ker H_{[2,3]} = \mathbb{C}^3 \otimes \mathcal{G}_2 \tag{2.44}$$

Let us quickly recall and prove a lemma from MAT201B:

Lemma: If A, B are non-negative definite operators on a complex Hilbert space $\mathcal{H}$, then

$$\ker(A+B) = \ker(A) \cap \ker(B) \tag{2.45}$$

Proof of Lemma :
($\supseteq$) Let $\psi \in \ker(A) \cap \ker(B)$. Then

$$(A+B)\psi = A\psi + B\psi = 0 \tag{2.46}$$

($\subseteq$) Let $\psi \in \ker(A+B)$, so $(A+B)\psi = 0$. The sum of two non-negative definite operators is again a non-negative definite operator, so $A+B$ is non-negative definite. We can thus take a square root $(A+B)^{1/2}$, and by looking at the eigenvalue zero and doing functional calculus with the square root function (zero has a unique square root), we see that $\ker(A+B) = \ker(A+B)^{1/2}$. Thus,

$$\begin{aligned}
0 = \left\|(A+B)\psi\right\|^2 = \left\|(A+B)^{1/2}\psi\right\|^2 &= \left\langle (A+B)^{1/2}\psi, (A+B)^{1/2}\psi \right\rangle \\
&= \left\langle \psi, (A+B)\psi \right\rangle \\
&= \langle \psi, A\psi\rangle + \langle \psi, B\psi\rangle \\
(A, B \geq 0) \qquad &\geq 0 + 0 \\
&= 0
\end{aligned} \tag{2.47}$$

And thus, since this is a complex Hilbert space, $\langle \psi, A\psi\rangle = \langle \psi, B\psi\rangle = 0$ implies $\psi \in \ker(A) \cap \ker(B)$. $\square$

Returning to the proof, since we can write

$$H_{[1,3]} = P^{(2)}_{1,2} + P^{(2)}_{2,3} = H_{[1,2]} + H_{[2,3]} \tag{2.48}$$

and since these are non-negative definite operators, we can apply the lemma and (2.44) to conclude that

$$\ker H_{[1,3]} = (\ker H_{[1,2]}) \cap (\ker H_{[2,3]}) = (\mathcal{G}_2 \otimes \mathbb{C}^3) \cap (\mathbb{C}^3 \otimes \mathcal{G}_2) \tag{2.49}$$

which is our desired intersection property.

Returning to our original goal for proving (2.42), we want to show $\dim\ker H_{[1,3]} \leq 4$. We will accomplish this by considering the decomposition of $\mathcal{G}_2 \otimes \mathbb{1}$ into irreducible representations of $\mathfrak{su}(2)$, and the fact that for any $\phi \in \ker H_{[1,3]}$, we have $\phi \in \mathcal{G}_2 \otimes \mathbb{C}^3$ and $\mathbb{1} \otimes P^{(2)}\phi = 0$ (which is a quick consequence of (2.42)).

$\mathcal{G}_2 = D^{(0)} \oplus D^{(1)}$, where $D^{(0)}$ is a singlet and $D^{(1)}$ is a triplet for $\mathfrak{su}(2)$. The tensor product distributes over direct sums, so

$$\begin{aligned}
\mathcal{G}_2 \otimes \mathbb{C}^3 = (D^{(0)} \oplus D^{(1)}) \otimes \mathbb{C}^3 &\cong (D^{(0)} \otimes \mathbb{C}^3) \oplus (D^{(1)} \otimes \mathbb{C}^3) \\
&\cong D^{(1)} \oplus (D^{(1)} \otimes D^{(1)}) \\
\textit{(Clebsch-Gordan)} \qquad &\cong D^{(1)} \oplus (D^{(2)} \oplus D^{(1)} \oplus D^{(0)}) \\
&\cong D^{(2)} \oplus (D^{(1)})^2 \oplus D^{(0)}
\end{aligned} \tag{2.50}$$

Or in words, we have one spin-2 representation, two triplets, and one singlet. We can once again do a Clebsch-Gordan decomposition, but note that since we are tensoring two different spaces this time, we now have that the $\widetilde{J}_3$ operator on $\mathcal{G}_2 \otimes \mathbb{C}^3$ that we will diagonalize is in fact given by

$$\widetilde{J}_3 = J_3^{(\mathcal{G}_2)} \otimes \mathbb{1} + \mathbb{1} \otimes J_3^{(\mathbb{C}^3)} \tag{2.51}$$

The highest weight eigenvalues for $J_3^{(\mathcal{G}_2)}$ and $J_3^{(\mathbb{C}^3)}$ are both 1, so we are looking for the unique eigenvector with highest weight $1+1=2$ eigenvalue of $\widetilde{J}_3$. This will be the highest weight eigenvector of the spin-2 subspace, just as before. Recall (1.24) and take the highest weight eigenvector of the triplet in $\mathcal{G}_2$, $|0,1\rangle - |1,0\rangle$, and the heighest weight eigenvector of $\mathbb{C}^3 \cong D^{(1)}$, $|1\rangle$. Tensor them together to get the highest weight eigenvector ξ_{22}:

$$\xi_{22} := (|0,1\rangle - |1,0\rangle) \otimes |1\rangle = |0,1,1\rangle - |1,0,1\rangle \tag{2.52}$$

But observe that $(\mathbb{1} \otimes P^{(2)})\xi_{22} \neq 0$:

$$\begin{aligned} (\mathbb{1} \otimes P^{(2)})\xi_{22} &= (\mathbb{1} \otimes P^{(2)})(|0,1,1\rangle - |1,0,1\rangle) \\ &= |0\rangle \otimes P^{(2)}(|1,1\rangle) - |1\rangle \otimes P^{(2)}(|0,1\rangle) \\ &= |0,1,1\rangle - |1\rangle \otimes P^{(2)}(|0,1\rangle) \\ &\neq 0 \end{aligned} \tag{2.53}$$

since $|1,1\rangle$ is in the $D^{(2)}$ subspace. Similarly, the highest weight vector for one of the spin-1 copies $D^{(1)}$ can be gained by tensoring a singlet eigenvector, $|1,-1\rangle - |0,0\rangle + |-1,1\rangle$, in $\mathcal{G}_2$ with an eigenvalue 1 eigenvector, $|1\rangle$, in $\mathbb{C}^3$. Then calling this vector ξ_{11}, we can compute

$$\begin{aligned} (\mathbb{1} \otimes P^{(2)})\xi_{11} &= (\mathbb{1} \otimes P^{(2)})(|1,-1,1\rangle - |0,0,1\rangle + |-1,1,1\rangle) \\ &= |1\rangle \otimes P^{(2)}|-1,1\rangle - |0\rangle \otimes P^{(2)}|0,1\rangle + |-1\rangle \otimes P^{(2)}|1,1\rangle \\ &= |1\rangle \otimes P^{(2)}|-1,1\rangle - |0\rangle \otimes P^{(2)}|0,1\rangle + |-1,1,1\rangle \\ &\neq 0 \end{aligned} \tag{2.54}$$

Taking the intersection property (2.49), the decomposition (2.50), and the computations (2.53) and (2.54) together, we can draw some conclusions. There is only one spin-2 representation $D^{(2)}$ in the subspace $\mathcal{G}_2 \otimes \mathbb{C}^3$, and so (2.53) implies that ξ_{22} is not in the ground state space and so the entire spin-2 representation $D^{(2)}$ is orthogonal to $\ker H_{[1,3]}$. Similarly, computation (2.54) shows that ξ_{11} is not in the ground state space, so at least one spin-1 representation $D^{(1)}$ is orthogonal to $\ker H_{[1,3]}$. This leaves only one copy of a spin-1 representation $D^{(1)}$ and one spin-0 representation $D^{(0)}$ to contain $\ker H_{[1,3]}$, by the intersection property (2.49). This all implies that

$$\dim \ker H_{[1,3]} = \dim(\mathcal{G}_2 \otimes \mathbb{C}^3) \cap (\mathbb{C}^3 \otimes \mathcal{G}_2) \leq \dim \mathcal{G}_2 \otimes \mathbb{C}^3 \leq \dim(D^{(0)}) + \dim(D^{(1)}) = 4 \tag{2.55}$$

Which is what we set out to show.

2.4.3 The Intersection Property

For the final step in the proof, we soup up the earlier intersection property. For the spaces $\mathcal{G}_n$. letting $\ell, r \geq 0, m \geq 2$,

$$(\mathcal{G}_{\ell+m} \otimes \mathcal{H}_{[1,r]}) \cap (\mathcal{H}_{[1,\ell]} \otimes \mathcal{G}_{m+r}) = \mathcal{G}_{\ell+m+r} \tag{2.56}$$

It will be convenient to use multi-indices $\mathbf{i} = (i_1, \ldots, i_\ell)$, $\mathbf{j} = (j_1, \ldots, j_m)$, $\mathbf{k} = (k_1, \ldots, k_r)$, and to let $v_{\mathbf{i}}$ denote the product $v_{i_\ell} \ldots v_{i_1}$, etc. Then, for $\phi \in (\mathcal{G}_{\ell+m} \otimes \mathcal{H}_{[1,r]}) \cap (\mathcal{H}_{[1,\ell]} \otimes \mathcal{G}_{m+r})$, we have $C_{\mathbf{i}}, D_{\mathbf{k}} \in M_2$ such that

$$\phi = \sum_{\mathbf{i}} |\mathbf{i}\rangle \otimes \psi^{(m+r)}(C_{\mathbf{i}}) = \sum_{\mathbf{k}} \psi^{\ell+m}(D_{\mathbf{k}}) \otimes |\mathbf{k}\rangle \tag{2.57}$$

We can expand this using our earlier definition of matrix product states (2.27):

$$0 = \left(\sum_{\mathbf{i}} |\mathbf{i}\rangle \otimes \left(\sum_{\mathbf{j},\mathbf{k}} \mathrm{Tr}[C_{\mathbf{i}} v_{\mathbf{k}} v_{\mathbf{j}}] \, |\mathbf{j},\mathbf{k}\rangle \right) \right) - \left(\sum_{\mathbf{k}} \left(\sum_{\mathbf{i},\mathbf{j}} \mathrm{Tr}[D_{\mathbf{k}} v_{\mathbf{j}} v_{\mathbf{i}}] \, |\mathbf{i},\mathbf{j}\rangle \right) \otimes |\mathbf{k}\rangle \right) \tag{2.58}$$

On the level of coefficients,

$$0 = \mathrm{Tr}[C_{\mathbf{i}} v_{\mathbf{k}} v_{\mathbf{j}}] - \mathrm{Tr}[D_{\mathbf{k}} v_{\mathbf{j}} v_{\mathbf{i}}] = \mathrm{Tr}[(C_{\mathbf{i}} v_{\mathbf{k}} - v_{\mathbf{i}} D_{\mathbf{k}}) v_{\mathbf{j}}] \tag{2.59}$$

This means that for all $\mathbf{i}, \mathbf{k}$, $\psi^{(m)}(C_{\mathbf{i}} v_{\mathbf{k}} - v_{\mathbf{i}} D_{\mathbf{k}}) = 0$. By assumption, $m \geq 2$, and so $\psi^{(m)}$ is injective and hence

$$C_{\mathbf{i}} v_{\mathbf{k}} - v_{\mathbf{i}} D_{\mathbf{k}} = 0, \quad \text{for all } \mathbf{i}, \mathbf{k} \tag{2.60}$$

Multiplying this relation on the left by $v_{\mathbf{i}}^*$ and summing over $\mathbf{i}$, we find

$$\left(\sum_{\mathbf{i}} v_{\mathbf{i}}^* C_{\mathbf{i}} \right) v_{\mathbf{k}} = \sum_{\mathbf{i}} (v_{\mathbf{i}}^* v_{\mathbf{i}}) D_{\mathbf{k}} \tag{2.61}$$

The isometry property of V means that $V^* V = \sum_{\mathbf{i}} v_{\mathbf{i}}^* v_{\mathbf{i}} = \mathbb{1}$, and therefore

$$D_{\mathbf{k}} = B v_{\mathbf{k}}, \quad \text{with } B = \sum_{\mathbf{i}} v_{\mathbf{i}}^* C_{\mathbf{i}} \tag{2.62}$$

Inserting this expression for $D_{\mathbf{k}}$ into (2.57) gives

$$\begin{aligned} \phi &= \sum_{\mathbf{k}} \psi^{(\ell+m)}(D_{\mathbf{k}}) \otimes |\mathbf{k}\rangle \\ &= \sum_{\mathbf{k}} \left(\sum_{\mathbf{i},\mathbf{j}} \mathrm{Tr}[D_{\mathbf{k}} v_{\mathbf{j}} v_{\mathbf{i}}] \, |\mathbf{i},\mathbf{j}\rangle \right) \otimes |\mathbf{k}\rangle \\ &= \sum_{\mathbf{k}} \left(\sum_{\mathbf{i},\mathbf{j}} \mathrm{Tr}[B v_{\mathbf{k}} v_{\mathbf{j}} v_{\mathbf{i}}] \, |\mathbf{i},\mathbf{j}\rangle \right) \otimes |\mathbf{k}\rangle \\ &= \psi^{(\ell+m+r)}(B) \end{aligned} \tag{2.63}$$

which completes the proof of the souped up intersection property (2.56).

We can finally finish the proof of the proposition by combining the intersection property (2.49) and the souped up intersection property (2.56), noting the conventions $\mathcal{H}_\emptyset = \mathbb{C}$ and $[a,b] = \emptyset$ if $a > b$.

$$\begin{aligned} \ker H_{[1,n]} &= \bigcap_{x=1}^{n-1} \mathcal{H}_{[1,x-1]} \otimes \mathcal{G}_2 \otimes \mathcal{H}_{[x+2,n]} \\ &= \bigcap_{x=1}^{n-2} \mathcal{H}_{[1,x-1]} \otimes (\mathcal{G}_2 \otimes \mathbb{C}^3 \cap \mathbb{C}^3 \otimes \mathcal{G}_2) \otimes \mathcal{H}_{[x+3,n]} \\ &= \bigcap_{x=1}^{n-2} \mathcal{H}_{[1,x-1]} \otimes \mathcal{G}_3 \otimes \mathcal{H}_{[x+3,n]} \\ &= \bigcap_{x=1}^{n-3} \mathcal{H}_{[1,x-1]} \otimes \mathcal{G}_4 \otimes \mathcal{H}_{[x+3,n]} \\ &\vdots \\ &= \mathcal{G}_n \end{aligned} \tag{2.64}$$

■

2.5 Valence Bond Solid

The intersection property (the souped up version, (2.56)) is 'visualized' in the Valence Bond Solid representation of the ground states of the AKLT chain. Recall the definition (2.13) of $\psi_{\alpha\beta}^{(n)}$ at the beginning of this chapter (relisted below for convenience)

$$\psi_{\alpha\beta}^{(n)} = (\mathbb{1}_3^{\otimes n} \otimes \langle\beta|)(\underbrace{\mathbb{1}_3 \otimes \cdots \otimes \mathbb{1}_3}_{n-1} \otimes V) \ldots (\mathbb{1} \otimes V)V\,|\alpha\rangle$$

Observe that, up to a normalization constant λ, the intertwining isometry $V : \mathbb{C}^2 \to \mathbb{C}^3 \otimes \mathbb{C}^2$ can be expressed as follows: for all $u \in \mathbb{C}^2$,

$$Vu = (P^+ \otimes \mathbb{1})(u \otimes \phi) \tag{2.65}$$

where $\phi \in \mathbb{C}^2 \otimes \mathbb{C}^2$ is the antisymmetric vector (i.e. the singlet state) and $P^+ : \mathbb{C}^2 \otimes \mathbb{C}^2 \to \mathbb{C}^3$ is the projection onto the symmetric states, which represent the triplet of spin-1 states. We can see this equality in two ways: first, by getting our hands dirty with bases and explicit matrices, and second, by working with intertwiners. We will do both.

Notice before continuing that we can express the right hand side of (2.65) as a linear map from $\mathbb{C}^2 \to \mathbb{C}^3 \otimes \mathbb{C}^2$:

$$(P^+ \otimes \mathbb{1})(u \otimes \phi) = (P^+ \otimes \mathbb{1})((\cdot) \otimes \phi)(u)$$

i.e. first map $u \mapsto u \otimes \phi$ (which is of course a linear embedding), then apply $P^+ \otimes \mathbb{1}$ to this.

First, the gritty explicit computation perspective. We will need to compute bases, and in the process we will see the mentioned antisymmetry of the singlet and symmetry of the triplet. The Clebsch-Gordan decomposition for $\mathbb{C}^2 \otimes \mathbb{C}^2 \cong D^{(1/2)} \otimes D^{(1/2)} \cong D^{(1)} \oplus D^{(0)}$ yields the following orthonormal bases (suppressing the underlying space spin-1/2 and writing $|1/2, 1/2; x, y\rangle = |x, y\rangle$):

$$\begin{aligned}
\text{Triplet } D^{(1)} &: \{|1/2, 1/2\rangle\,,\ \frac{1}{\sqrt{2}}\Big(|1/2, -1/2\rangle + |-1/2, 1/2\rangle\Big)\,,\ |-1/2, -1/2\rangle\} \\
&= \{|\uparrow\uparrow\rangle\,,\ \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)\,,\ |\downarrow\downarrow\rangle\} \\
&=: \{|+\rangle\,, |0\rangle\,, |-\rangle\} \\
\text{Singlet } D^{(0)} &: \{\frac{1}{\sqrt{2}}\Big(|1/2, -1/2\rangle - |-1/2, 1/2\rangle\Big)\} \\
&= \{\frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)\} \\
&=: \{|s\rangle\}
\end{aligned} \tag{2.66}$$

Now to verify claim (2.65). Choose the standard basis $\{|\uparrow\rangle\,, |\downarrow\rangle\}$ for $\mathbb{C}^2$, and let $u := u_+\,|\uparrow\rangle + u_-\,|\downarrow\rangle \in \mathbb{C}^2$. The antisymmetric vector $\phi \in \mathbb{C}^2 \otimes \mathbb{C}^2$ can be expressed in the standard tensor product basis as $\phi = |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle$. Then, starting from the right side of (2.65),

$$\begin{aligned}
(P^+ \otimes \mathbb{1})(u \otimes \phi) &= (P^+ \otimes \mathbb{1})\left[(u_+\,|\uparrow\rangle + u_-\,|\downarrow\rangle) \otimes (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)\right] \\
&= (P^+ \otimes \mathbb{1})\left[u_+\,|\uparrow\uparrow\downarrow\rangle + u_-\,|\downarrow\uparrow\downarrow\rangle - u_+\,|\uparrow\downarrow\uparrow\rangle - u_-\,|\downarrow\downarrow\uparrow\rangle\right] \\
(\textit{change to } (1.24) \textit{ basis}) \qquad &= (P^+ \otimes \mathbb{1})\left[u_+\,|+\downarrow\rangle + u_-\left(\frac{1}{\sqrt{2}}\,|0\downarrow\rangle - \frac{1}{\sqrt{2}}\,|s\downarrow\rangle\right) - u_+\left(\frac{1}{\sqrt{2}}\,|0\uparrow\rangle + \frac{1}{\sqrt{2}}\,|s\uparrow\rangle\right) - u_-\,|-\uparrow\rangle\right] \\
(P^+ \textit{ projects onto } D^{(1)}) \qquad &= u_+\,|+\downarrow\rangle + \frac{1}{\sqrt{2}}u_-\,|0\downarrow\rangle - \frac{1}{\sqrt{2}}u_+\,|0\uparrow\rangle - u_-\,|-\uparrow\rangle \\
&= u_+\,|+\downarrow\rangle - \frac{1}{\sqrt{2}}u_+\,|0\uparrow\rangle + \frac{1}{\sqrt{2}}u_-\,|0\downarrow\rangle - u_-\,|-\uparrow\rangle
\end{aligned} \tag{2.67}$$

So, if we choose the standard basis $\{|\uparrow\rangle, |\downarrow\rangle\}$ for $\mathbb{C}^2$ and a basis for $\mathbb{C}^3 \otimes \mathbb{C}^2$ by tensoring the symmetric state basis (which represents a triplet of spin-1 states) with the standard basis for $\mathbb{C}^2$,

$$|+\uparrow\rangle = \begin{bmatrix}1\\0\\0\\0\\0\\0\end{bmatrix}, |+\downarrow\rangle = \begin{bmatrix}0\\1\\0\\0\\0\\0\end{bmatrix}, |0\uparrow\rangle = \begin{bmatrix}0\\0\\1\\0\\0\\0\end{bmatrix}, |0\downarrow\rangle = \begin{bmatrix}0\\0\\0\\1\\0\\0\end{bmatrix}, |-\uparrow\rangle = \begin{bmatrix}0\\0\\0\\0\\1\\0\end{bmatrix}, |-\downarrow\rangle = \begin{bmatrix}0\\0\\0\\0\\0\\1\end{bmatrix} \tag{2.68}$$

Then the matrix for $(P^+ \otimes \mathbb{1})((\cdot)\otimes\phi) : \mathbb{C}^2 \to \mathbb{C}^3 \otimes \mathbb{C}^2$ is given by

$$(P^+ \otimes \mathbb{1})((\cdot)\otimes\phi) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \\ 0 & -1 \\ 0 & 0 \end{bmatrix} \tag{2.69}$$

So in fact, looking at our earlier explicit expression for V (2.22), we see that this matrix is in fact a scalar multiple of V:

$$(P^+ \otimes \mathbb{1})((\cdot)\otimes\phi) = \sqrt{3}V \tag{2.70}$$

which is what was claimed for (2.65).

Second, the intertwiner perspective. Notice firstly that the orthogonal projection onto the symmetric states $P^+ : \mathbb{C}^2 \otimes \mathbb{C}^2 \to \mathbb{C}^3$ in fact intertwines the representations $D^{(1/2)} \otimes D^{(1/2)}$ and $D^{(1)}$ (which can be seen by Clebsch-Gordon decomposition $D^{(1/2)} \otimes D^{(1/2)} \cong D^{(1)} \oplus D^{(0)}$), just as V intertwined $D^{(1/2)}$ with $D^{(1)} \otimes D^{(1/2)}$ (2.10).

$$P^+(D^{(1/2)} \otimes D^{(1/2)}) = D^{(1)}P^+ \tag{2.71}$$

Then we have the following:

$$\begin{aligned}
\left[(P^+ \otimes \mathbb{1})((\cdot)\otimes\phi)\right]\left(D^{(1/2)}u\right) &= (P^+ \otimes \mathbb{1})((D^{(1/2)}u)\otimes\phi) \\
(a) \qquad &= (P^+ \otimes \mathbb{1})\left((D^{(1/2)}u) \otimes \left((D^{(1/2)} \otimes D^{(1/2)})\lambda\phi\right)\right) \\
&= \lambda(P^+ \otimes \mathbb{1})\left(D^{(1/2)} \otimes D^{(1/2)} \otimes D^{(1/2)}\right)(u \otimes \phi) \\
(\text{by } 2.71) \qquad &= \lambda\left(D^{(1)}P^+ \otimes D^{(1/2)}\right)(u\otimes\phi) \\
&= \lambda\left[(D^{(1)} \otimes D^{(1/2)})\right](P^+ \otimes \mathbb{1})((\cdot)\otimes\phi)(u)
\end{aligned} \tag{2.72}$$

where (a) follows since ϕ is the singlet state and thus invariant under the action of $D^{(1/2)} \otimes D^{(1/2)} \cong D^{(1)} \oplus D^{(0)}$ up to a complex phase $\lambda \in \mathbb{C}$. This tells us that if we renormalize ϕ such that $\lambda = 1$, the linear map $(P^+ \otimes \mathbb{1})((\cdot)\otimes\phi)$ in fact obeys the same intertwiner property as V (2.10). Between two irreducible representations, intertwiners are unique up to complex phase by Schur's Lemma. If we restrict $(P^+ \otimes \mathbb{1})((\cdot)\otimes\phi)$ to the invariant subspace $D^{(1)} \subseteq D^{(1/2)} \otimes D^{(1/2)}$, then since it is clearly not the 0 map, it must be a scalar multiple of V, confirming our earlier explicit computation (2.70).

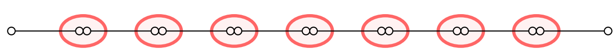
Now we can finally move on to show the expression claimed in the notes. We will start by writing the very first expression for the vector $\psi^{(n)}_{\alpha\beta}$, (2.13). Let $|\alpha\rangle, |\beta\rangle$ be eigenstates of the third Pauli matrix σ^3, so either $|\uparrow\rangle$ or $|\downarrow\rangle$.

$$
\begin{aligned}
\psi^{(n)}_{\alpha\beta} &= (\mathbb{1}_3^{\otimes n} \otimes \langle\beta|)(\underbrace{\mathbb{1}_3 \otimes \cdots \otimes \mathbb{1}_3}_{n-1} \otimes V)\dots(\mathbb{1}\otimes V)V\,|\alpha\rangle \\
&= (\mathbb{1}_3^{\otimes n} \otimes \langle\beta|)(\underbrace{\mathbb{1}_3 \otimes \cdots \otimes \mathbb{1}_3}_{n-1} \otimes V)\dots(\mathbb{1}\otimes V)(P^+\otimes\mathbb{1})(|\alpha\rangle\otimes\phi) \\
&= (\mathbb{1}_3^{\otimes n} \otimes \langle\beta|)(\underbrace{\mathbb{1}_3 \otimes \cdots \otimes \mathbb{1}_3}_{n-1} \otimes V)\dots(P^+\otimes V)(|\alpha\rangle\otimes\phi) \\
&= (\mathbb{1}_3^{\otimes n} \otimes \langle\beta|)(\underbrace{\mathbb{1}_3 \otimes \cdots \otimes \mathbb{1}_3}_{n-1} \otimes V)\dots(P^+\otimes P^+\otimes\mathbb{1})(|\alpha\rangle\otimes\phi\otimes\phi) \\
&\ \,\vdots \\
&= \Big[\underbrace{P^+\otimes\cdots\otimes P^+}_{n}\otimes\langle\beta|\Big]\Big[|\alpha\rangle\otimes\underbrace{\phi\otimes\cdots\otimes\phi}_{n}\Big] \\
&= \Big[\underbrace{P^+\otimes\cdots\otimes P^+}_{n}\Big]\Big[\underbrace{\mathbb{1}_4\otimes\cdots\otimes\mathbb{1}_4}_{n-1}\otimes\mathbb{1}_2\otimes(\mathbb{1}_2\otimes\langle\beta|)\Big]\Big[|\alpha\rangle\otimes\underbrace{\phi\otimes\cdots\otimes\phi}_{n}\Big] \\
(*)\qquad &= \pm\Big[\underbrace{P^+\otimes\cdots\otimes P^+}_{n}\Big]\Big[|\alpha\rangle\otimes\underbrace{\phi\otimes\cdots\otimes\phi}_{n-1}\otimes|-\beta\rangle\Big] \\
&= VBS
\end{aligned}
\tag{2.73}
$$

where $(*)$ follows by an explicit computation (since $|\beta\rangle$ is an eigenstate of σ^3):

$$
\begin{aligned}
\big[\mathbb{1}_2\otimes\langle\uparrow|\big]\,\phi &= (\mathbb{1}_2\otimes\langle\uparrow|)\big(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\big) \\
&= -\,|\downarrow\rangle \\
\big[\mathbb{1}_2\otimes\langle\downarrow|\big]\,\phi &= (\mathbb{1}_2\otimes\langle\downarrow|)\big(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\big) \\
&= |\uparrow\rangle
\end{aligned}
$$

This final expression for Valence Bond States gives rise to the following picture:



Here's the idea: each small white circle corresponds to a spin-1/2 particle. At any of the middle sites of the lattice, we can realize a spin-1 particle by tensoring together two spin-1/2 particles $D^{(1/2)}\otimes D^{(1/2)}$ and then projecting onto the subspace of symmetric states $P^+ : D^{(1/2)}\otimes D^{(1/2)} \to D^{(1)}$. This process is depicted by the red ovals surrounding pairs of spin-1/2 particles.

Then, between neighboring spin-1 sites, our spin-1/2 particles are "tied" together in singlet states, which are the antisymmetric vectors ϕ, depicted by lines between the spin-1/2 particles.

Finally, we are left with two spin-1/2 particles at the edges of the chain: on the left, $|\alpha\rangle$, and on the right, $|-\beta\rangle$. The vectors $\psi^{(n)}_{\alpha\beta}$ are fully determined by $|\alpha\rangle, |\beta\rangle$, and so pictorially, these ground states of the finite chain are fully describable once we know the spin-1/2 particles at both edges.

2.6 Finitely Correlated States and the Thermodynamic Limit

We now study correlation functions and the thermodynamic limit. We can rewrite (using cyclicity of trace) the definition (2.13) of $\psi^{(n)}_{\alpha\beta}$ as

$$
\psi^{(n)}_{\alpha\beta} = \sum_{i_1,\dots,i_n} \langle\beta|\, v_{i_n}\dots v_{i_1}\,|\alpha\rangle\,|i_1,\dots,i_n\rangle \tag{2.74}
$$

and so for $A_1, \ldots, A_n \in M_3$:

$$
\begin{aligned}
&\left\langle \psi_{\alpha\beta}^{(n)} \middle| A_1 \otimes \cdots \otimes A_n \middle| \psi_{\alpha\beta}^{(n)} \right\rangle \\
\textit{(just matrix mult.)} \qquad &= \sum_{\substack{i_1,\ldots,i_n \\ j_1,\ldots j_n}} \overline{\langle\beta| v_{i_n} \ldots v_{i_1} |\alpha\rangle} (A_1)_{i_1 j_1} \ldots (A_n)_{i_n j_n} \langle\beta| v_{j_n} \ldots v_{j_1} |\alpha\rangle \\
&= \sum_{\substack{i_1,\ldots,i_n \\ j_1,\ldots j_n}} (A_1)_{i_1 j_1} \ldots (A_n)_{i_n j_n} \langle\alpha| v_{i_1}^* \ldots v_{i_n}^* |\beta\rangle \langle\beta| v_{j_n} \ldots v_{j_1} |\alpha\rangle \\
&= \sum_{\substack{i_1,\ldots,i_{n-1} \\ j_1,\ldots j_{n-1}}} (A_1)_{i_1 j_1} \ldots (A_n)_{i_{n-1} j_{n-1}} \\
&\qquad \times \langle\alpha| v_{i_1}^* \ldots v_{i_{n-1}}^* \left[\sum_{i_n, j_n} (A_n)_{i_n j_n} v_{i_n}^* |\beta\rangle \langle\beta| v_{j_n} \right] v_{j_{n-1}} \ldots v_{j_1} |\alpha\rangle
\end{aligned} \tag{2.75}
$$

To shed light on the bracketed part of the equation, we define for all $A \in M_3$ a map $\mathbb{E}_A : M_2 \to M_2$ by

$$
\mathbb{E}_A(B) = \sum_{ij} v_i^* A_{ij} B v_j = V^*(A \otimes B)V \tag{2.76}
$$

One may find the block matrix representation of this useful:

$$
V^*(A \otimes B)V = \begin{bmatrix} v_1^* & v_0^* & v_{-1}^* \end{bmatrix} \begin{bmatrix} A_{11}B & A_{10}B & A_{1,-1}B \\ A_{01}B & A_{00}B & A_{0,-1}B \\ A_{-1,1}B & A_{-1,0}B & A_{-1,-1}B \end{bmatrix} \begin{bmatrix} v_1 \\ v_0 \\ v_{-1} \end{bmatrix}
$$

In terms of these maps we can write the expectations of general tensor product observables above in a more compact form:

$$
\begin{aligned}
&\left\langle \psi_{\alpha\beta}^{(n)} \middle| A_1 \otimes \cdots \otimes A_n \middle| \psi_{\alpha\beta}^{(n)} \right\rangle \\
&\quad = \sum_{\substack{i_1,\ldots,i_{n-1} \\ j_1,\ldots j_{n-1}}} (A_1)_{i_1 j_1} \ldots (A_n)_{i_{n-1} j_{n-1}} \langle\alpha| v_{i_1}^* \ldots v_{i_{n-1}}^* [\mathbb{E}_{A_n}(|\beta\rangle\langle\beta|)] v_{j_{n-1}} \ldots v_{j_1} |\alpha\rangle \\
&\quad = \langle\alpha| \mathbb{E}_{A_1} \circ \cdots \circ \mathbb{E}_{A_n}(|\beta\rangle\langle\beta|) |\alpha\rangle
\end{aligned} \tag{2.77}
$$

This expression makes the calculation of the thermodynamic limit very transparent. Adding $\ell + 1$ sites to the left and r to the right of the interval $[1, n]$ gives the following expression for the expectation of $\mathbb{1}^{\otimes \ell} \otimes A_1 \otimes \cdots \otimes A_n \otimes \mathbb{1}^{\otimes r} \in \mathcal{A}_{[-\ell+1, n+r]}$ in the vector state $\psi_{\alpha\beta}^{(\ell+n+r)}$:

$$
\begin{aligned}
\left\langle \psi_{\alpha\beta}^{(\ell+n+r)} \middle| A \middle| \psi_{\alpha\beta}^{(\ell+n+r)} \right\rangle &= \langle\alpha| \mathbb{E}_{\mathbb{1}}^{\ell} \circ \mathbb{E}_{A_1} \circ \cdots \circ \mathbb{E}_{A_n} \circ \mathbb{E}_{\mathbb{1}}^{r}(|\beta\rangle\langle\beta|) |\alpha\rangle \\
&= \mathrm{Tr}\, |\alpha\rangle\langle\alpha| \mathbb{E}_{\mathbb{1}}^{\ell} \circ \mathbb{E}_{A_1} \circ \cdots \circ \mathbb{E}_{A_n} \circ \mathbb{E}_{\mathbb{1}}^{r}(|\beta\rangle\langle\beta|) \\
&= \mathrm{Tr} (\mathbb{E}_{\mathbb{1}}^{T})^{\ell} (|\alpha\rangle\langle\alpha|)^{T} \circ \mathbb{E}_{A_1} \circ \cdots \circ \mathbb{E}_{A_n} \circ \mathbb{E}_{\mathbb{1}}^{r}(|\beta\rangle\langle\beta|) \\
&= \mathrm{Tr} (\mathbb{E}_{\mathbb{1}}^{T})^{\ell} (|\alpha\rangle\langle\alpha|) \circ \mathbb{E}_{A_1} \circ \cdots \circ \mathbb{E}_{A_n} \circ \mathbb{E}_{\mathbb{1}}^{r}(|\beta\rangle\langle\beta|)
\end{aligned} \tag{2.78}
$$

where for the third line, $(\mathbb{E}_{\mathbb{1}}^{T})$ denotes the transpose of $\mathbb{E}_{\mathbb{1}}$ with respect to the Hilbert-Schmidt inner product on M_2.

The map $\mathbb{E}_{\mathbb{1}}$ is called the *transfer operator* and its spectral properties control the limits $\lim_{\ell\to\infty}$ and $\lim_{r\to\infty}$. Using for the isometry V the matrix form (2.22) or the intertwiner equality (2.10) we can verify the following diagonalization of $\mathbb{E}_{\mathbb{1}}$ with definition 2.76

$$\begin{aligned} \mathbb{E}_{\mathbb{1}}(\mathbb{1}) &= V^*(\mathbb{1} \otimes \mathbb{1})V \\ &= \mathbb{1} \\ \mathbb{E}_{\mathbb{1}}(\sigma^k) &= \sum_{ij} v_i^* \mathbb{1}_{ij} \sigma^1 v_j \\ &= v_1^* \sigma^k v_1 + v_0^* \sigma^k v_0 + v_{-1}^* \sigma^k v_{-1} \\ &= -\frac{1}{3}\sigma^k \end{aligned} \tag{2.79}$$

Now, with respect to the scaled Hilbert-Schmidt inner product $\frac{1}{2}\langle \cdot, \cdot \rangle_{HS}$, the matrices $\{\mathbb{1}, \sigma^1, \sigma^2, \sigma^3\}$ form an orthonormal basis for M_2. So for any $B \in M_2$, we can write

$$B = \frac{1}{2}(\text{Tr}B)\mathbb{1} + \frac{1}{2}\sum_{i=1}^{3}(\text{Tr}B\sigma^i)\sigma^i$$

which allows us to compute

$$\begin{aligned} \mathbb{E}_{\mathbb{1}}(B) &= \frac{1}{2}(\text{Tr}B)\mathbb{E}_{\mathbb{1}}(\mathbb{1}) + \frac{1}{2}\sum_{i=1}^{3}(\text{Tr}B\sigma^i)\mathbb{E}_{\mathbb{1}}(\sigma^i) \\ &= \frac{1}{2}(\text{Tr}B)\mathbb{1} - \frac{1}{3}\left[\sum_{i=1}^{3}(\text{Tr}B\sigma^i)\sigma^i\right] \\ &= \frac{1}{2}(\text{Tr}B)\mathbb{1} - \frac{1}{3}[B - \frac{1}{2}(\text{Tr}B)\mathbb{1}] \end{aligned}$$

and therefore

$$\begin{aligned} \mathbb{E}_{\mathbb{1}}(B) &= \frac{1}{2}(\text{Tr}B)\mathbb{1} - \frac{1}{3}[B - \frac{1}{2}(\text{Tr}B)\mathbb{1}] \\ \mathbb{E}_{\mathbb{1}}^2(B) &= \frac{1}{2}(\text{Tr}B)\mathbb{1} - \frac{1}{3}\left[\left(\frac{1}{2}(\text{Tr}B)\mathbb{1} - \frac{1}{3}[B - \frac{1}{2}(\text{Tr}B)\mathbb{1}]\right) - \frac{1}{2}(\text{Tr}B)\mathbb{1}\right] \\ &= \frac{1}{2}(\text{Tr}B)\mathbb{1} + \left(-\frac{1}{3}\right)^2[B - \frac{1}{2}(\text{Tr}B)\mathbb{1}] \\ &\vdots \\ \mathbb{E}_{\mathbb{1}}^p(B) &= \frac{1}{2}(\text{Tr}B)\mathbb{1} + \left(-\frac{1}{3}\right)^p[B - \frac{1}{2}(\text{Tr}B)\mathbb{1}] \end{aligned} \tag{2.80}$$

Note that for the special case of $B = |\beta\rangle\langle\beta|$, this expression also gives us that $\mathbb{E}_{\mathbb{1}}^T(|\beta\rangle\langle\beta|) = \mathbb{E}_{\mathbb{1}}(|\beta\rangle\langle\beta|)$. This in turn allows us to compute the limit

$$\begin{aligned} &\lim_{\ell\to\infty, r\to\infty} \langle\alpha| \mathbb{E}_{\mathbb{1}}^\ell \circ \mathbb{E}_{A_1} \circ \cdots \circ \mathbb{E}_{A_n} \circ \mathbb{E}_{\mathbb{1}}^r(|\beta\rangle\langle\beta|)|\alpha\rangle \\ &= \lim_{\ell\to\infty, r\to\infty} \text{Tr}\,(\mathbb{E}_{\mathbb{1}}^T)^\ell(|\alpha\rangle\langle\alpha|) \circ \mathbb{E}_{A_1} \circ \cdots \circ \mathbb{E}_{A_n} \circ (\mathbb{E}_{\mathbb{1}})^r(|\beta\rangle\langle\beta|) \\ &= \lim_{\ell\to\infty, r\to\infty} \text{Tr}\,\mathbb{E}_{\mathbb{1}}^\ell(|\alpha\rangle\langle\alpha|) \circ \mathbb{E}_{A_1} \circ \cdots \circ \mathbb{E}_{A_n} \circ (\mathbb{E}_{\mathbb{1}})^r(|\beta\rangle\langle\beta|) \\ &= \frac{\|\alpha\|^2\|\beta\|^2}{4}\text{Tr}\,\mathbb{1} \circ \mathbb{E}_{A_1} \circ \cdots \circ \mathbb{E}_{A_n} \circ \mathbb{1} \\ &= \frac{\|\alpha\|^2\|\beta\|^2}{4}\text{Tr}\,\mathbb{E}_{A_1} \circ \cdots \circ \mathbb{E}_{A_n} \\ &=: \omega(A_1 \otimes \cdots \otimes A_n) \end{aligned} \tag{2.81}$$

where ω is the unique translation invariant pure state on $\mathcal{A}_{\mathbb{Z}}$ uniquely determined by the above expression for simple tensor observables.

Define $Q : M_2 \to M_2$ by

$$Q(B) = \frac{1}{2}(\mathrm{Tr}B)\mathbb{1} \tag{2.82}$$

Again using computation (2.80), we estimate for $B \in M_2$

$$\begin{aligned} \left\|(\mathbb{E}_{\mathbb{1}}^{p} - Q)B\right\| &= \left\|\frac{1}{2}(\mathrm{Tr}B)\mathbb{1} + \left(-\frac{1}{3}\right)^{p}\left[B - \frac{1}{2}(\mathrm{Tr}B)\mathbb{1}\right] - \frac{1}{2}(\mathrm{Tr}B)\mathbb{1}\right\| \\ &= \left(\frac{1}{3}\right)^{p}\left\|B - \frac{1}{2}(\mathrm{Tr}B)\mathbb{1}\right\| \\ &\leq \left(\frac{1}{3}\right)^{p} 2\|B\| \end{aligned} \tag{2.83}$$

where the last line is an application of the triangle inequality and the following inequality:

> *Lemma:* Let $B \in M_n$. Then the following inequality (where $\|\cdot\|$ denotes the Hilbert-Schmidt norm) holds:
>
> $$\frac{1}{n}\left\|(\mathrm{Tr}B)\mathbb{1}\right\|^2 \leq \|B\|^2$$
>
> *Proof of Lemma* : Recall that the trace is the sum of the eigenvalues of B, call them $\lambda_1, \ldots, \lambda_n$. Then
>
> $$\left\|(\mathrm{Tr}B)\mathbb{1}\right\|^2 = n\left(\sum_{i=1}^{n}\lambda_i\right)^2 \leq n\sum_{i=1}^{n}|\lambda_i|^2 = n\|B\|^2$$
>
> □

Putting together the expression for the state ω from (2.81) for the special case where $A_2 = \cdots = A_{n-1} = \mathbb{1}$, and using the estimate (2.83), we have an estimate for the two-point correlation function of ω:

$$\begin{aligned} \left|\omega(A_1 \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes A_n) - \omega(A_1)\omega(A_n)\right| &= \left|\frac{1}{2}\mathrm{Tr}\mathbb{E}_{A_1} \circ (\mathbb{E}_{\mathbb{1}}^{n-2} - Q) \circ \mathbb{E}_{A_n}\right| \\ &\leq C\|\mathbb{E}_{A_1}\|\left\|\mathbb{E}_{\mathbb{1}}^{n-2} - Q\right\|\|\mathbb{E}_{A_n}\| \\ &\leq \|A_1\|\|A_n\|\frac{C}{3^n} \end{aligned} \tag{2.84}$$

Thus, we have shown exponential decay of correlations in the state ω, or in other words, the correlation length of this state is $\ln 3$. Next, we show that ω is the unique zero-energy ground state of the AKLT chain.

2.6.1 Proposition 11.2

Proposition 11.2, Nachtergaele and Sims:

ω defined by (2.81) is the unique state on $\mathcal{A}_{\mathbb{Z}}$ such that $\omega(P^{(2)}_{x,x+1}) = 0$, for all $x \in \mathbb{Z}$.

PROOF:

Any state η on $\mathcal{A}_{\mathbb{Z}}$ is uniquely determined by its restrictions to the subalgebras $\mathcal{A}_{[a,b]}, a < b$.

Let $\rho_{[a,b]}$ denote the density matrices of η restricted to $\mathcal{A}_{[a,b]}$. Notice that $\eta(P^{(2)}_{x,x+1}) = 0$, for $x = a, \ldots, b-1$. This tells us that $\mathrm{ran}\,\rho_{[a,b]} \subseteq \mathcal{G}_{b-a+1}$: to see why, recall that any density matrix ρ is expressible as a convex combination of pure states $|\psi_i\rangle\langle\psi_i|, \psi_i \in \mathcal{H}$: i.e., $0 \leq c_i \leq 1$ and

$$\rho = \sum_i c_i |\psi_i\rangle\langle\psi_i|, \qquad \sum_i c_i = 1$$

So every vector in the range of our operator $\rho_{[a,b]}$ has the property that it has zero expectation for the operator $P^{(2)}$, which means it is in $\ker H_{[a,b]}$. By Proposition 11.1, $\mathcal{G}_{[a,b]} = \ker H_{[a,b]}$.

We then have, for all $a_1 < b_1 \in \mathbb{Z}, A_{a_1}, \ldots, A_{b_1} \in M_3$,

$$\begin{aligned} &\eta(A_{a_1} \otimes \cdots \otimes A_{b_1}) \\ &\quad = \lim_{\substack{a \to -\infty \\ b \to \infty}} \mathrm{Tr} \rho_{[a,b]} \mathbb{1}_{a,a_1-1} \otimes A_{a_1} \otimes \cdots \otimes A_{b_1} \otimes \mathbb{1}_{b_1+1,b} \\ (2.81) &\quad = \omega(A_{a_1} \otimes \cdots \otimes A_{b_1}) \end{aligned}$$

Which is what we wanted to show. ■

The so-called Haldane phase is characterized by three properties. The AKLT model satisfies all of these properties, and we have thus far proven the first two:

- Unique ground state
- Finite correlation length
- Non-vanishing spectral gap above the ground state

Our next goal is to prove that the final property in fact holds for the AKLT model by moving to a more general class of quantum spin chains: those with a finite number of ground states that are of the MPS form.

3 References (sloppily enumerated, but present)

1. Nachtergaele and Sims' "Introduction to Quantum Spin Systems"
2. Amanda Young's presentation https://www.math.ucdavis.edu/~bxn/AKLT_Model-2.pdf
3. https://www2.ph.ed.ac.uk/~ldeldebb/docs/QM/lect8.pdf
4. Brian Hall's Lie Groups, Lie Algebras, and Representations (2015)