Convergence to the Perron Projection

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Summer 2020

Lemma 0.1. *Recall: Linear algebra (Dimension of eigenspaces)*
For every eigenvalue λ of a matrix A, the adjoint A^* has $\overline{\lambda}$ as an eigenvalue, and each corresponding eigenspace has the same dimension.

Proof. A matrix and its adjoint have the same rank, by e.g. Jordan Canonical Form or by this cute proof. Then, by rank-nullity theorem,

$$\dim \operatorname{ran}\,(A - \lambda \mathbb{1}) = \dim \operatorname{ran}\,(A^* - \overline{\lambda}\mathbb{1}) \Leftrightarrow \dim \ker(A - \lambda \mathbb{1}) = \dim \ker(A^* - \overline{\lambda}\mathbb{1})$$

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Theorem 0.2. *Convergence to Perron Projection*
Let $\mathbb{E} : \mathbb{C}^k \to \mathbb{C}^k$ be a linear operator with spectral radius $r(E) \leq 1$ and with 1 as a simple eigenvalue. Further, assume that $\mathbb{E}$ has trivial peripheral spectrum, i.e. that there are no other eigenvalues on the unit circle. Then there exists $C > 0$ and $\lambda \in (0, 1)$ such that the following inequality holds:

$$\left\| \mathbb{E}^n - |v\rangle\langle w| \right\| \leq C\lambda^n$$

where $\|\cdot\|$ is operator norm, $|v\rangle$ and $|w\rangle$ are right and left eigenvectors of $\mathbb{E}$ of eigenvalue 1, normalized so that $\langle w, v\rangle = 1$.

Proof. We already know that since $\mathbb{E}$ has 1 as a simple eigenvalue, the left and right eigenspaces are spanned by eigenvectors of eigenvalue 1 by Lemma (0.1). So in equations,

$$\mathbb{E}|v\rangle = |v\rangle, \qquad \langle w|\mathbb{E} = \langle w|$$

Notice that we necessarily have $\langle w, v\rangle \neq 0$, and we can choose these vectors such that $\langle w, v\rangle = 1$. This is because we can write $\mathbb{E}$ in Jordan canonical form $\mathbb{E} = SJS^{-1}$ where, by simplicity of the eigenvalue 1, J can be chosen to have the block diagonal form:

$$J = \begin{bmatrix} 1 & 0 \\ 0 & * \end{bmatrix}$$

So, choosing the left and right eigenvectors $\langle w| = \langle e_1|S$ and $|v\rangle = S^{-1}|e_1\rangle$, where $e_1 = [1, 0, \ldots, 0]^T$, we have

$$\langle w, v\rangle = \langle w|\mathbb{E}|v\rangle = \langle e_1|S(S^{-1}JS)S^{-1}|e_1\rangle = \langle e_1|J|e_1\rangle = 1$$

This then guarantees that the map $|v\rangle\langle w|$ is a rank-1 projection onto the 1-dimensional eigenspace for eigenvalue 1:

$$(|v\rangle\langle w|)^2 = |v\rangle\langle w, v\rangle\langle w| = |v\rangle\langle w|$$

and it further commutes with $\mathbb{E}$ and enjoys the following relation, using our eigenvector equations:

$$|v\rangle\langle w|\mathbb{E} = |v\rangle\langle w| = \mathbb{E}|v\rangle\langle w|$$

Let us call this projection $P := |v\rangle\langle w|$ and rewrite the above:

$$P\mathbb{E} = P = \mathbb{E}P \tag{0.1}$$

This allows us to easily take powers:

$$(\mathbb{E}-P)^n = \mathbb{E}^n - P$$

Now, we will show that the spectral radius of the operator $r(\mathbb{E}-P) < 1$. We will then provide two different proofs (in the claims) that both yield the desired inequality from this fact. Observe that since $P, \mathbb{E}$ commute, we can block diagonalize, and using (0.1) multiple times, we have:

$$\begin{aligned}\mathbb{E} &= \mathbb{E}P + \mathbb{E}(\mathbb{1}-P)\\ &= P\mathbb{E}P + (\mathbb{1}-P)\mathbb{E}(\mathbb{1}-P)\\ &= P + (\mathbb{1}-P)\mathbb{E}(\mathbb{1}-P)\end{aligned}$$

and so

$$\mathbb{E}(\mathbb{1}-P) = \mathbb{E}-P = (\mathbb{1}-P)\mathbb{E}(\mathbb{1}-P)$$

Suppose λ is an eigenvalue of $\mathbb{E}(\mathbb{1}-P)$ with eigenvector x, so

$$\mathbb{E}(\mathbb{1}-P)x = \lambda x, \qquad x \neq 0$$

Applying $(\mathbb{1}-P)$ to both sides and commuting things,

$$(1-P)\mathbb{E}(1-P)x = \mathbb{E}(1-P)x = \lambda(1-P)x$$

So $(1-P)x$ is an eigenvector of $\mathbb{E}$ of eigenvalue λ.

If $(1-P)x \neq 0$, then since P is a rank 1 projection onto the 1-eigenspace $\mathrm{span}(|v\rangle)$ of $\mathbb{E}$, we have that $(1-P)x \notin \mathrm{span}(|v\rangle)$. Since $\mathbb{E}$ has trivial peripheral spectrum, $|\lambda| < 1$.

If $(1-P)x = 0$, then $x = Px$, so $x \in \mathrm{span}(|v\rangle)$. By projection, $\lambda = 0$.

Thus, $r(\mathbb{E}-P) < 1$, as desired.

Claim: Let $\mathbb{E}$ and P be as above. Then there exists $C > 0$ and $\lambda \in (0,1)$ such that the following inequality holds:

$$\|\mathbb{E}^n - P\| \leq C\lambda^n$$

Proof: *(Proof using Gelfand's formula)* Pick $1 > \lambda > r(\mathbb{E}-P)$. Gelfand's formula gives us that

$$\begin{aligned}\lambda &> r(\mathbb{E}-P)\\ &= \lim_{n\to\infty} \left\|(\mathbb{E}-P)^n\right\|^{1/n}\\ \text{(0.1)} \quad &= \lim_{n\to\infty} \|\mathbb{E}^n - P\|^{1/n}\end{aligned}$$

So, picking a sufficiently large $C > 0$ to control the first few terms of the sequence, we have the following desired inequality for all n:

$$\|\mathbb{E}^n - P\| \leq C\lambda^n$$

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Proof: *(Proof using Jordan Canonical Form)* Note the following formula for the n^{th} powers of a $\ell \times \ell$ Jordan block with eigenvalue α:

$$J_\ell(\alpha)^n = \begin{bmatrix} \alpha^n & \binom{n}{1}\alpha^{n-1} & \binom{n}{2}\alpha^{n-2} & \cdots & \cdots & \binom{n}{\ell-1}\alpha^{n-\ell+1} \\ & \alpha^n & \binom{n}{1}\alpha^{n-1} & \cdots & \cdots & \binom{n}{\ell-2}\alpha^{n-\ell+2} \\ & & \ddots & \ddots & \vdots & \vdots \\ & & & \ddots & \ddots & \vdots \\ & & & & \alpha^n & \binom{n}{1}\alpha^{n-1} \\ & & & & & \alpha^n \end{bmatrix}$$

$$= \alpha^n \begin{bmatrix} 1 & \binom{n}{1}\alpha^{-1} & \binom{n}{2}\alpha^{-2} & \cdots & \cdots & \binom{n}{\ell-1}\alpha^{-\ell+1} \\ & 1 & \binom{n}{1}\alpha^{-1} & \cdots & \cdots & \binom{n}{\ell-2}\alpha^{-\ell+2} \\ & & \ddots & \ddots & \vdots & \vdots \\ & & & \ddots & \ddots & \vdots \\ & & & & 1 & \binom{n}{1}\alpha^{-1} \\ & & & & & 1 \end{bmatrix}$$

Since $r(\mathbb{E} - P) < 1$, the Jordan Canonical Form of $\mathbb{E} - P$ has that every Jordan block has eigenvalue $\alpha < 1$. Note that the largest binomial term $\binom{n}{l-1}$ grows at a polynomial rate in n, $O(n^{l-1})$, so the growth in operator norm of the matrix above (after factoring out α^n is at most polynomial in n. Thus, using (0.1), we can choose a constant $C > 0$ and a $1 > \lambda > r(\mathbb{E} - P)$ such that

$$\begin{aligned} \|\mathbb{E}^n - P\| &= \left\|(\mathbb{E} - P)^n\right\| \\ &= \left\| \sum_{\alpha \in \sigma(\mathbb{E}-P)} J(\alpha)^n \right\| \\ &\le \sum_{\alpha \in \sigma(\mathbb{E}-P)} \left\|J(\alpha)^n\right\| \\ &\le C\lambda^n \end{aligned}$$

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