Final, Part 2

MAT 21D, Ragone

2021 Summer II

1. Each part of this exam is **40 minutes**. At the end of the 40 minutes, you will have 10 minutes to scan & upload your exam.
2. This exam is open note, open homework, and open book. You may **not** use any external electronic or online resource, nor may you work with others.
3. The final is worth **135 points** total, **68 points** for part 1 and **67 points** for part 2. Where a question has multiple parts, the point breakdown has been listed.
4. You must show **all** your work. Answers without work will receive no credit. Be specific when citing theorems.
5. You may either use physical pen & paper, or an electronic note-taking app. Answer each question on a new page. Please label continuing work by the number and "cont." (e.g., if problem 3 had 3 pages, you could write "3", "3, cont.", "3, cont. 2")
6. Clearly indicate which page goes with which problem. Upload all your pages to GradeScope, and correctly match each question with the pages.
7. Stay logged in to the Zoom room until you have uploaded your exam, in case you need technical assistance. **You do not need to keep your camera on.** If you are disconnected from the Zoom room for any reason, do not worry; message me through Canvas, or send me an e-mail, and continue to work on your midterm until I respond, or until time is up.

This exam is a chance to show off what you have learned so far. If you know how to do part of a question, write down what you know. Good luck!!

Problem 4 (37 points)

Consider the surface S_{top} in $\mathbb{R}^3$ given by the cylinder $x^2 + y^2 = 4$ bounded on the bottom by the plane $z = 0$ and on top by the plane $z = 1$. Combine this with the surface S_{bottom} given by the half-sphere $x^2 + y^2 + z^2 = 4$ when $z \leq 0$ to form a single closed surface $S = S_{top} \cup S_{bottom}$, oriented outwards. Call the region bounded by this surface D (i.e. $S = \partial D$).

(a) Sketch this region. (**7 points**)

(b) Let $\vec{F} = -x\hat{\imath} - y\hat{\jmath} - z\hat{k}$. Using a theorem from class, compute the surface integral

$$\oiint_S \vec{F} \cdot \vec{n}\, d\sigma$$

Hint: The volume of a sphere is $\frac{4}{3}\pi r^3$. (**9 points**)

(c) (**16 points**)

(i) Let $\vec{G}, \vec{H}$ be smooth vector fields on $\mathbb{R}^3$ such that $\vec{G} = \nabla \times \vec{H}$. Using a theorem from class, write down (but do not compute) an integral which calculates $\iint_{S_{top}} \vec{G} \cdot \vec{n}\, d\sigma$ in terms of an area integral depending upon $\vec{G}$ in the xy plane (you may use polar coordinates if you prefer).

(ii) Let $\vec{H} = -y\hat{\imath} + x\hat{\jmath} + e^z\hat{k}$, and let $\vec{G} = \nabla \times \vec{H}$. Using the expression you found in part (i), Calculate $\iint_{S_{top}} \vec{G} \cdot \vec{n}\, d\sigma$.

(iii) Using part (ii) and a theorem from class, calculate $\iint_{S_{bottom}} \vec{G} \cdot \vec{n}\, d\sigma$.

(d) Compute the surface integral

$$\oiint_S \left(\nabla \times (\nabla(e^{\arctan(xy)}))\right) \cdot \vec{n}\, d\sigma$$

(**5 points**)

Problem 5 (10 points)

In probability theory, we often encounter various "stretched and moved" Gaussian distributions, thanks to differing standard deviations and means. Thankfully, we really only need to know the integral we calculated in PS 2, listed below, and how coordinate transformations work.

$$\pi = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)}\, dxdy$$

Use a change of coordinates to calculate the following integral:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-((x-2)^2+4y^2)}\, dxdy$$

Problem 6 (20 points)

Decide whether the below statements are true or false. In either case, you must reasonably argue why (note that counterexamples are logical reasons). Credit will only be awarded for proper reasoning. If you use a theorem, you must state what conditions are satisfied!

(a) (T/F): (**5 points**) Let $\vec{F}(x, y, z)$ be a vector field on $\mathbb{R}^3$ and let $\vec{r}(t), a \leq t \leq b$ be a parameterization of an oriented curve C. Then the following integral outputs a scalar:

$$\int_a^b \vec{F} \left|\vec{r'}(t)\right|\, dt$$

(b) (T/F): (**5 points**) Consider the oriented curve C in $\mathbb{R}^2$ given by a straight line connecting the point $(0, 0)$ to $(1, 1)$, and consider two parameterizations $\vec{r}_1, \vec{r}_2$ of this line:

$$\vec{r}_1(t) = \langle 0, 0\rangle + t\langle 1, 1\rangle; \qquad 0 \leq t \leq 1$$
$$\vec{r}_2(t) = \langle 0, 0\rangle + t\langle 2, 2\rangle; \qquad 0 \leq t \leq \frac{1}{2}$$

Then the following line integrals are equal:

$$\int_0^1 \left|\vec{r_1'}(t)\right|\, dt = \int_0^{\frac{1}{2}} \left|\vec{r_2'}(t)\right|\, dt$$

(c) (T/F): (**5 points**) Suppose we have a smooth scalar function $f(x, y)$ defined on $\mathbb{R}^2$. Let C be a closed level curve of $f(x, y)$ given by the equation $f(x, y) = c$ for a constant c. Then

$$\oint_C \nabla f \cdot d\vec{r} = 0$$

(d) (T/F): (**5 points**) Suppose we have a smooth scalar function $f(x, y)$ defined on $\mathbb{R}^2$. Let C be a level curve of $f(x, y)$ given by the equation $f(x, y) = c$ for a constant c. Then

$$\oint_C \nabla f \cdot \vec{n}\, ds = 0$$